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INTEGRATION BY PARTS FOR THE HENSTOCK–STIELTJES INTEGRALS

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ABSTRACT. In this paper, we investigate continuity of

$$F(x) = (H)\int_a^x f dG$$

and Henstock-Stieltjes integrability of product of two functions and obtain the formula of integration by parts for the Henstock-Stieltjes integral.

1. Introduction

Henstock integral is a generalization of Riemann integral. This integral includes the Lebesgue integral. The definition of the Henstock integral is very similar to the definition of the Riemann integral as we will see. On this situation, it is very natural that we consider the Stieltjes type integral for the Henstock integral. This is the Henstock-Stieltjes integral.

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In order to define the Henstock-Stieltjes(H-S) integral, it is necessary to specify the terms that are allowed in the Riemann-Stieltjes sums. We begin by looking at certain types of tagged partitions. Pay close attention to the notation and terminology.

DEFINITION 1.1. Let $\delta(x)$ be a positive function defined on $[a,b]$. A *tagged interval* $(x,[c,d])$ consists of an interval $[c,d] \subset [a,b]$ and a point $x \in [c,d]$. The tagged interval $(x,[c,d])$ is *subordinate* to δ if $[c,d] \subset (x-\delta(x), x+\delta(x))$. The letter P will be used to denote finite collections of nonoverlapping tagged intervals. Let $P = \{(x_i,[c_i,d_i]) : 1 \leq i \leq n\}$ be such a collection in $[a,b]$. We adopt the following terminology,

(a) The points $\{x_i\}$ are the tags of P and the intervals $[c_i,d_i]$ are the intervals of P.

(b) If $(x_i,[c_i,d_i])$ is subordinate to δ for each i, then P is subordinate to δ.

(c) If P is subordinate to δ and $[a,b] = \cup_{i=1}^{n}[c_i,d_i]$, then P is a tagged partition of $[a,b]$ that is subordinate to δ.

Let $P = \{(x_i,[c_i,d_i]) : 1 \leq i \leq n\}$ be a finite collection of nonoverlapping tagged intervals in $[a,b]$, let $f : [a,b] \to R$ and let F be a function defined on the subintervals of $[a,b]$. We will use the following notations,

$$f^G(P) = \sum_{i=1}^{n} f(x_i)(G(d_i) - G(c_i)),$$

$$F(P) = \sum_{i=1}^{n} F([c_i,d_i]) = \sum_{i=1}^{n}(F(d_i) - F(c_i))$$

and $F(x) = \int_a^x f dG$ will always be treated as a function of intervals when defined on tagged partition of $[a,b]$, that is,

$$F([c,d]) = F(d) - F(c) = \int_a^d f dG - \int_a^c f dG.$$

DEFINITION 1.2. Let f and G be two finite valued functions on $[a, b]$. Then f is *Henstock-Stieltjes integrable* with respect to G on $[a, b]$ if there exists a real number L with the following property: for every $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such thatvert $|f^G(P) - L| < \epsilon$ whenever P is a tagged partition of $[a, b]$ that is subordinate to δ. We denote the integral as $L = (H)\int_a^b f dG$. The function f is Henstock-Stieltjes integrable with respect to G on a measurable set $E \subset [a, b]$ if $f\chi_E$ is Henstock-Stieltjes integrable with respect to G on [a,b].

We use the following lemma frequently to prove theorems on the H-S integral.

LEMMA 1.3. (Henstock-Stieltjes Lemma) *Let f be H-S integrable with respect to G, and let $F(x) = \int_a^x f dG$ for each $x \in [a, b]$, and let $\epsilon > 0$. Suppose that δ is a positive function on $[a, b]$ such that $|f^G(P) - F(P)| < \epsilon$ whenever P is a tagged partition of $[a, b]$ that is subordinate to δ. If $P_0 = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ is subordinate to δ, then*

$$|f^G(P_0) - F(P_0)| \leq \epsilon,$$

$$\sum_{i=1}^{n} |f(x_i)(G(d_i) - G(c_i)) - (F(d_i) - F(c_i))| \leq 2\epsilon.$$

We need the next lemma to obtaine the formula of integration by parts.

LEMMA 1.4. *Let $\{\alpha_k : 1 \leq k \leq n\}$ and $\{\beta_k : 1 \leq k \leq n\}$ be two finite collections of real numbers. Then*

$$\sum_{k=1}^{n} \alpha_k \beta_k = \sum_{k=1}^{n-1} \sum_{i=1}^{k} \alpha_i (\beta_k - \beta_k + 1) + \sum_{i=1}^{n} \alpha_i \beta_n.$$

2. Main Results

THEOREM 2.1. *Let $f : [a,b] \longrightarrow R$ is H-S integrable with respect to G on $[a,b]$, where G is continuous on $[a,b]$. And let $F(x) = \int_a^x f dG$ for each x in $[a,b]$. Then the function F is continuous on $[a,b]$.*

Proof. Let $\epsilon > 0$ be given. Since f is H-S integrable on $[a,b]$, there exists a positive function δ_1 on $[a,b]$ such that $|f^G(P) - F(P)| < \epsilon/3$ whenever P is a tagged partition of $[a,b]$ that is subordinate δ_1. For each y in $[a,b]$, since G is continuous on $[a,b]$, there exists a $\delta_2(y) > 0$ such that $|G(x) - G(y)| < \epsilon/3(|f(y)|+1)$, whenever$|x-y| < \delta_2(y)$. Let $\delta = \min\{\delta_1, \delta_2\}$. By the Henstock-Stieltjes Lemma,

$$\begin{aligned} |F(x) - F(y)| \leq & |F(x) - F(y) - f(y)(G(x) - G(y))| \\ & + |f(y)(G(x) - G(y))| \\ \leq & \frac{2}{3}\epsilon + \frac{\epsilon|f(y)|}{3(|f(y)|+1)} \leq \epsilon \end{aligned}$$

whenever $|x - y| < \delta(y)$. Therefore, the function F is continuous at each $y \in [a,b]$. □

We use this theorem to prove the following theorem.

THEOREM 2.2. *Let $f : [a,b] \longrightarrow R$ is H-S integrable with respect to H on $[a,b]$, where H is continuous on $[a,b]$ and let $F(x) = \int_a^x f dH$ for each x in $[a,b]$. If $G : [a,b] \longrightarrow R$ is of bounded variation on $[a,b]$, then fG is H-S integrable on $[a,b]$ with respect to H and*

$$\int_a^b fGdH = F(b)G(b) - \int_a^b FdG.$$

Proof. Let $\epsilon > 0$. Since f is H-S integrable on $[a,b]$ with respect to H, there exists a positive function δ_1 defined on $[a,b]$ such that

$$|f^H(P) - \int_a^b fdH| < \epsilon,$$

where P is a tagged partition of $[a,b]$ that is subordinate to δ_1. By Theorem 2.1, F is continuous on $[a,b]$. Since F is Riemann-Stieltjes integrable with respect to G on $[a,b]$, there exists $\eta > 0$ such that

$$|\sum_{i=1}^{n} F(c_i)(G(x_i) - G(x_i - 1)) - \int_a^b FdG| < \epsilon$$

where $P = \{(c_i, [x_i - 1, x_i]) : 1 \leq i \leq n\}$ is a tagged partition of $[a,b]$ with $\|P\| < \eta$. Define δ on $[a,b]$ by

$$\delta(x) = \begin{cases} \min\{\delta_1, \eta/2, x-a, b-x\}, & \text{if } x \in (a,b), \\ \min\{\delta_1(x), \eta/2\}, & \text{if } x = a \text{ or } b. \end{cases}$$

Let $P = \{(c_k, [x_k - 1, x_k]) : 1 \leq k \leq n\}$ be a tagged partition of $[a,b]$ that is subordinate to δ and assume that each tag occurs only once. Note that $c_1 = a$ and that $c_n = b$. Using Lemma 1.4 and the Henstock-Stieltjes Lemma, we obtain

$$\begin{aligned}
&|\sum_{k=1}^{n} f(c_k)G(c_k)H_k - (F(b)G(b) - \int_a^b FdG)| \\
&= |\sum_{k=1}^{n-1}\sum_{i=1}^{k} f(c_i)H_iG_k + \sum_{i=1}^{n} f(c_i)H_iG(c_n) - (F(b)G(b) - \int_a^b FdG)| \\
&\leq \sum_{k=1}^{n-1} |G_k|\, |\sum_{i=1}^{k} f(c_i)H_i - F(x_k)| + |\sum_{k=1}^{n-1} F(x_k)G_k - \int_a^b FdG| \\
&\quad + |G(b)||\sum_{i=1}^{n} f(c_i)H_i - F(b)| \\
&< \epsilon V(G,[a,b]) + \epsilon + \epsilon|G(b)|,
\end{aligned}$$

where $H_i = H(x_i) - H(x_i - 1)$, $G_k = G(c_k) - G(c_k + 1)$ for $1 \leq i, k \leq n$ and $V(G,[a,b])$ is variation of G on $[a,b]$. Hence the function fG is H-S integrable on $[a,b]$ with respect to H on $[a,b]$ and

$$\int_a^b fGdH = F(b)G(b) - \int_a^b FdG.$$

This completes the proof. □

REFERENCES

1. R. A. Górdon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., 1994.
2. I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar publishing Co, 1964.
3. W. Pfeffer, *A note on the Generallized Riemann integral*, proc. Amer. Math. Soc **103** (1988), 1161-1166.
4. S. Saks, *Theory of the integrals*, Hafner, New York, 1937.
5. W. Rudin, *Principles of mathematical analysis, Third edition*, McGraw-Hill, New York.
6. Y. Kubota, *Remarks on the Henstock integral of Stieltjes type*, Bull. Fac. Sci.(Ser. A) **17** (1985), 25-29.

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