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INTEGRAL REPRESENTATION OF SOME BLOCH TYPE FUNCTIONS IN $\mathbb{C}^n$

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ABSTRACT. Let B be the open unit ball in the complex space $\mathbb{C}^n$. A holomorphic function $f : B \to C$ which satisfies $\sup\{(1- < z, z >) \| \nabla_z f \| \, |z \in B\} < +\infty$ is called Bloch type function. In this paper, we will find some integral representation of Bloch type functions.

1. Introduction

Throughout this paper, $\mathbb{C}$ will denote the complex field, and $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$. Let D be the open unit disk in $\mathbb{C}$. Then the Bloch space of D consists of analytic functions f on D such that

$$\sup\{(1 - |z|^2)|f'(z)| \mid z \in D\} < +\infty.$$

Bloch functions in the Bloch space of D are well known and have been studied by many authors [See 1].

In this paper, we will consider some functions on the open unit ball B in complex space $\mathbb{C}^n$. Specifically, we let $\mathcal{B}$ denote the linear space of all holomorphic functions $f : B \to C$ which satisfy

$$\sup\{(1- < z, z >) \| \nabla_z f \| \, |z \in B\} < +\infty.$$

In [5], Timoney showed that the linear space $\mathcal{B}$ is a Banach space under the norm

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$$\| f \|_{\mathcal{B}} = |f(0)| + \sup\{(1- < z, z >) \| \nabla_z f \| \mid z \in B\}.$$

The purpose of this paper is to find some integral representation of Bloch type functions in $\mathcal{B}$.

2. Integral representation

Let $a \in B$ and let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the subspace generated by a, which is given by

$$P_a z = \begin{cases} \dfrac{< z, a >}{< a, a >} a, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases}$$

Let $Q_a = I - P_a$. Define φ_a on B by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - \| a \|^2} Q_a z}{1- < z, a >}.$$

THEOREM 2.1. *Let $a \in B$. Then*

(i) *φ_a is a biholomorphic mapping of B onto itself.*

(ii) *$\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a(\varphi_a(z)) = z$.*

(iii) *For all $z, w \in \overline{B}$, we have*

$$1- < \varphi_a(z), \varphi_a(w) > = \frac{(1 - \| a \|^2)(1- < z, w >)}{(1- < z, a >)(1- < a, w >)},$$

$$1 - \| \varphi_a(z) \|^2 = \frac{(1 - \| a \|^2)(1 - \| z \|^2)}{1 - | < z, a > |^2}.$$

Proof. See [3, Theorem 2.2.2]. □

THEOREM 2.2. *Let ψ be a biholomorphic mapping of B onto itself and $a = \psi^{-1}(0)$. The determinant $J_R\psi$ of the real Jacobian matrix of ψ satisfies the following identity:*

$$|J_R\psi(z)|^2 = \left(\frac{1 - \| a \|^2}{|1- < z, a > |^2} \right)^{n+1} = \left(\frac{1 - \| \psi(z) \|^2}{1- \| z \|^2} \right)^{n+1}.$$

Proof. See [3, Theorem 2.2.6]. □

Let ν be the Lebesgue measure on $\mathbb{C}^n$ normalized by $\nu(B) = 1$. We let σ be the rotation invariant positive Borel measure on S for which $\sigma(S) = 1$. Here, S is the boundary of B. The term "rotation - invariant"refers to the invariant porperty of σ i.e., $\sigma(\rho E) = \sigma(E)$ for every Borel set $E \subset S$ and for every rotation $\rho \in O(2n)$ which is the group of all isometries of $\mathbb{R}^{2n}$ that fix the origin. The measure ν and σ are related by the formula

$$\int_{\mathbb{C}^n} f d\nu = 2n \int_0^\infty r^{2n-1} dr \int_S f(r\zeta) d\sigma(\zeta).$$

The measure μ is the weighted Lebesgue measure:

$$d\mu = (n+1)(1 - \| z \|^2) d\nu(z).$$

LEMMA 2.3. *If $f \in L^1_\mu(B) \cap H(B)$ where $H(B)$ is the set of holomorphic functions on B, then $f(0) = \int_B f(z) d\mu(z)$.*

Proof. Since $f \in H(B)$, by the mean value theorem

$$f(0) = \int_S f(r\zeta) d\sigma(\zeta), \; 0 < r < 1. \tag{1}$$

By integrating both sides of (1) with respect to the measure $2n(1 - r^2) r^{2n-1} dr$ over $[0, 1]$, we have

$$2n \int_0^1 \int_S f(r\zeta)(1 - r^2) r^{2n-1} d\sigma(\zeta) dr = f(0)(n+1)^{-1}.$$

Namely,

$$f(0) = (n+1) \int_B f(z)(1 - \| z \|^2) d\nu(z) = \int_B f(z) d\mu(z).$$

□

THEOREM 2.4. *If f is analytic on B and* $\int_B(1-\| z \|^2)|f(z)|d\nu(z) < \infty$ *, then*

$$f(z) = (n+1)\int_B \frac{1-\| w \|^2}{(1-<z,w>)^{n+2}} f(w)d\nu(w).$$

Proof. By Lemma 2.3,

$$f(0) = (n+1)\int_B f(w)(1-<w,w>)d\nu(w).$$

Replace f by $f \circ \varphi_z$ and using Theorem 2.1, 2.2, then

$$\begin{aligned} f(z) =& (f \circ \varphi_z)(0) \\ =& (n+1)\int_B (f \circ \varphi_z)(w)(1-<w,w>)d\nu(w) \\ =& (n+1)\int_B f(w)(1-|\varphi_z(w)|^2)\left(\frac{(1-||z||^2)}{|1-<w,z>|^2}\right)^{n+1} d\nu(w) \\ =& (n+1)\int_B f(w)\frac{(1-\| z \|^2)^{n+2}(1-\| w \|^2)}{|1-<w,z>|^{2n+4}}d\nu(w) \\ =& (n+1)(1-\| z \|^2)^{n+2}\int_B f(w)\frac{(1-\| w \|^2)}{|1-<w,z>|^{2n+4}}d\nu(w). \end{aligned}$$

Replace $f(w)$ again by $f(w)(1-<w,z>)^{n+2}$, then

$$\begin{aligned} & f(z)(1-\| z \|^2)^{n+2} \\ &= (n+1)(1-\| z \|^2)^{n+2}\int_B f(w)\frac{(1-\| w \|^2)}{(1-<z,w>)^{n+2}}d\nu(w). \end{aligned}$$

So, we have

$$f(z) = (n+1)\int_B \frac{(1-\| w \|^2)}{(1-<z,w>)^{n+2}} f(w)d\nu(w).$$

□

THEOREM 2.5. *Suppose that $z \in B$ and $f \in \mathcal{B}$. Then*

$$f(z) = f(0) + \int_B \frac{(1-\| w \|^2)\nabla_w f \cdot z}{< z, w > (1- < z, w >)^{n+1}} d\nu(w),$$

where $\nabla_w f \cdot z = \sum_{i=1}^{n} \frac{\partial f(w)}{\partial w_i} \cdot z_i$.

Proof. By Theorem 2.4,

$$\nabla_{tz} f = (n+1) \int_B \frac{1-\| w \|^2}{(1- < tz, w >)^{n+2}} \nabla_w f d\nu(w).$$

Taking the line integral from 0 to z, we get

$$\begin{aligned} f(z) - f(0) &= \int_0^1 \nabla_{tz} f \cdot z dt \\ &= \int_0^1 (n+1) \int_B \frac{(1-\| w \|^2)\nabla_w f \cdot z}{(1- < tz, w >)^{n+2}} d\nu(w) dt \\ &=(n+1) \int_B (1-\| w \|^2)\nabla_w f \cdot z \int_0^1 \frac{1}{(1-t < z, w >)^{n+2}} dt\, d\nu(w) \\ &= \int_B (1-\| w \|^2)\nabla_w f \cdot z \frac{1}{< z, w >} \left[\frac{1}{(1- < z, w >)^{n+1}} - 1 \right] d\nu(w) \\ &= \int_B (1-\| w \|^2)\nabla_w f \cdot z \frac{1}{< z, w > (1- < z, w >)^{n+1}} d\nu(w) \\ &\quad - \int_B (1-\| w \|^2)\nabla_w f \cdot z \frac{1}{< z, w >} d\nu(w). \end{aligned}$$

It is easy to see (using Taylor expansion) that

$$\int_B (1-\| w \|^2)\nabla_w f \cdot z \frac{1}{< z, w >} d\nu(w) = 0.$$

Thus we obtain the desired result as follows:

$$f(z) - f(0) = \int_B \frac{(1-\| w \|^2)\nabla_w f \cdot z}{< z, w > (1- < z, w >)^{n+1}} d\nu(w).$$

□

References

1. S. Axler, *The Bergman spaces, the Bloch space and commutators of multiplication operators*, Duke Math. J. **53** (1986), 315-332.
2. K.T. Hahn, *Holomorphic mappings of the hyperbolic space into the complex Euclidean space and Bloch theorem*, Canadian J. Math. **27** (1975), 446-458.
3. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag (1980).
4. E.M. Stein, *Boundary behavior of holomorphic functions of several complex variables*, Mathematical Notes, Princeton Univ. Press, Princeton.
5. R.M.Timoney, *Bloch functions of several variables*, J. Bull. London Math. Soc. **12** (1980), 241-267.
6. K.H. Zhu, *Hankel Operators on the Bergman spaces of bounded symmetric domains*, Trans. Amer. Math. Soc. **324** (1991), 707-730.

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