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# SOME PROPERTIES OF QUOTIENT FUZZY NORMED LINEAR SPACES

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ABSTRACT. The main goal of this paper is to investigate some properties of a quotient fuzzy seminorm $\rho_q$ induced by a fuzzy seminorm $\rho = \chi_{B_{\|\cdot\|}}$ on a normed linear space $X$.

## 1. Introduction

Katsaras and Liu [4] introduced the notions of fuzzy linear spaces and fuzzy topological linear spaces. These ideas were modified by Katsaras [2] and in [3], where Katsaras defined the notions of fuzzy seminorms and fuzzy norms on a linear space. In [7], Rhie, Choi and Kim introduced the notions of the fuzzy $\alpha$-Cauchy sequence and the fuzzy completeness, and studied some of related properties of fuzzy normed linear spaces.

The purpose of this paper is to prove some properties of fuzzy Cauchy sequences, and to investigate the relation between a quotient fuzzy seminorm $\rho = \chi_{B_{\|\cdot\|}}$ and a fuzzy seminorm $\chi_{B_{|||\cdot|||}}$ on $X/W$, where $W$ is a closed subspace of uniformly convex Banach space $X$.

Let $X$ be a linear space over the field $K$ ($R$ or $C$) throughout this paper. A fuzzy set in $X$ is an element of the set $I^X$ of all functions from $X$ into the unit interval $I$. In general, fuzzy subsets of $X$ are denoted by Greek letters. $\chi_A$ denotes the characteristic function of

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the set $A$. By a *fuzzy point* $\mu$ we mean a fuzzy subset $\mu : X \to [0,1]$ such that

$$\mu(z) = \begin{cases} \alpha, & \text{if } z = x, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$. We usually denote the fuzzy point with support $x$ and value $\alpha$ by $(x, \alpha)$. $\vee$ and $\wedge$ are used for the supremum and infimum of the family of fuzzy sets respectively. And supp$\mu = \{x \in X \mid \mu(x) > 0\}$ is the support of $\mu$.

DEFINITION 1.1.([4]) If $\mu \in I^X$ and $t \in K$, then

$$(t\mu)(x) = \begin{cases} \mu(x/t), & \text{if } t \neq 0, \\ 0, & \text{if } t = 0 \text{ and } x \neq 0, \\ \vee_{y \in X} \mu(y), & \text{if } t = 0 \text{ and } x = 0. \end{cases}$$

DEFINITION 1.2.([4]) $\mu \in I^X$ is said to be:

(a) *convex* if $t\mu + (1-t)\mu \leq \mu$ for each $t \in [0,1]$,
(b) *balanced* if $t\mu \leq \mu$ for each $t \in K$ with $|t| \leq 1$,
(c) *absolutely convex* if $\mu$ is convex and balanced,
(d) *absorbing* if $\vee_{t>0} t\mu(x) = 1$ for all $x \in X$.

DEFINITION 1.3.([3]) A *fuzzy seminorm* $\rho$ on $X$ is a fuzzy subset of $X$ which satisfies the following three conditions:

(a) $\rho$ is convex,
(b) $\rho$ is balanced,
(c) $\rho$ is absorbing.

If, in addition, a fuzzy seminorm $\rho$ satisfies the condition

(d) $\wedge_{t>0} t\rho(x) = 0$ for $x \neq 0$ in $X$,

then $\rho$ is called a *fuzzy norm*.

DEFINITION 1.4.([6]) Let $\rho$ be a fuzzy seminorm on $X$. Then $P_\epsilon : X \to R_+$ is defined by $P_\epsilon(x) = \wedge\{t > 0 | t\rho(x) > \epsilon\}$ for each $\epsilon \in (0,1)$ and $P_{\alpha^-} : X \to R_+$ defined by $P_{\alpha^-}(x) = \vee_{\epsilon < \alpha} P_\epsilon(x)$ for every $x \in X$.

THEOREM 1.5. ([5, Theorem 3.2]) *For each* $\epsilon \in (0,1)$, *every* $P_\epsilon$ *is a seminorm on* $X$. *Further* $P_\epsilon$ *is a norm on* $X$ *for each* $\epsilon \in (0,1)$ *if and only if* $\rho$ *is a fuzzy norm on* $X$.

DEFINITION 1.6.([6]) Let $(X,\tau)$ be a fuzzy topological space, $\{\mu_n = (x_n, \alpha_n)\}$ a sequence of fuzzy points in $X$ and $\mu = (x, \alpha)$ a fuzzy point in $X$. We say that $\{\mu_n\}$ converges to $\mu$,written as $\mu_n \to \mu$ if and only if given a neighbourhood $N$ of $\mu$ there exists a positive integer $M$ such that $n \geq M$ implies $\mu_n \leq N$.

THEOREM 1.7. ([6, Theorem 3.2]) *Let* $(X,\rho)$ *be a fuzzy normed linear space and* $\{\mu_n = (x_n, \alpha_n)\}$ *be a sequence of fuzzy points in* $X$. *Then* $\mu_n \to \mu = (x,\alpha)$ *if and only if for every* $t > 0$, *there exists* $M \in Z^+$ *such that for all* $n \geq M, \alpha_n \leq \alpha$ *and* $P_{\alpha_n^-}(x_n - x) < t$.

DEFINITION 1.8.([7]) Let $\alpha \in (0,1)$. A sequence of fuzzy points $\{\mu_n = (x_n, \alpha_n)\}$ is said to be a *fuzzy* $\alpha$*-Cauchy sequence* in a fuzzy normed linear space $(X,\rho)$ if for each zero neighborhood $N$ with $N(0) > \alpha$, there exists a positive integer $M$ such that $n,m \geq M$ implies $\mu_n - \mu_m = (x_n - x_m, \alpha_n - \alpha_m) \leq N$. A fuzzy normed linear space $(X,\rho)$ is said to be *fuzzy* $\alpha$*-complete* if every fuzzy $\alpha$-Cauchy sequence $\{\mu_n\}$ converges to a fuzzy point $\mu = (x,\alpha)$. $(X,\rho)$ is said to be *fuzzy complete* if it is fuzzy $\alpha$-complete for every $\alpha \in (0,1)$. And a fuzzy complete normed linear space is called a *fuzzy Banach space.*

THEOREM 1.9. ([7, Theorem 3.2]) *Let* $(X,\rho)$ *be a fuzzy normed linear space and* $\alpha \in (0,1)$. *Then* $\{(x_n, \alpha_n)\}$ *is a fuzzy* $\alpha$*-Cauchy sequence if and only if for each* $t > 0$, *there exists a positive integer* $M$ *such that* $m,n \geq M$ *implies* $\alpha_n \wedge \alpha_m \leq \alpha$ *and* $P_{(\alpha_n \wedge \alpha_m)^-}(x_n - x_m) < t$.

## 2. Main results

Throughout this section, $X$ denotes a normed linear space.

LEMMA 2.1. *Let* $(X, \|\cdot\|)$ *be a normed linear space. If* $\rho = \chi_B$, *where* $B = \{\, x \in X \mid \|x\| \le 1 \,\}$, *then for each* $\epsilon \in (0,1)$, $P_\epsilon(x) = \|x\|$ *for all* $x \in X$.

*Proof.* For all $x \in X$, $\epsilon \in (0,1)$,

$$\begin{aligned} P_\epsilon(x) &= \wedge\{\, s > 0 \mid s\rho(x) > \epsilon \,\} \\ &= \wedge\{\, s > 0 \mid \rho(x/s) > \epsilon \,\} \\ &= \wedge\{\, s > 0 \mid \rho(x/s) = 1 \,\} \quad \text{as } \rho = \chi_B \\ &= \wedge\{\, s > 0 \mid \|x/s\| \le 1 \,\} \quad \text{as } x/s \in B \\ &= \wedge\{\, s > 0 \mid \|x\| \le s\} \\ &= \|x\|. \end{aligned}$$

□

LEMMA 2.2. *Let* $(X, \chi_B)$ *be a fuzzy normed linear space and* $\{(x_n, \alpha_n)\}$ *be a sequence of fuzzy points in* $X$ *and* $(x, \alpha)$ *a fuzzy point in* $X$. *If* $\{(x_n, \alpha_n)\}$ *converges to* $(x, \alpha)$, *then* $\{x_n\}$ *converges to* $x$.

*Proof.* Since $\{(x_n, \alpha_n)\}$ converges to $(x, \alpha)$, we have that for every $t > 0$, there exists $M \in Z^+$ such that for all $n \ge M$, $\alpha_n \le \alpha$ and $P_{\alpha_n^-}(x_n - x) < t$ by Theorem 1.7. And since $P_{\alpha_n^-}(x_n - x) = \|x_n - x\|$, it deduce that $\{x_n\}$ converges to $x$. □

THEOREM 2.3. *Let* $\{(x_n, \alpha_n)\}$ *be a fuzzy* $\alpha$*-Cauchy sequence in a fuzzy normed linear space* $(X, \chi_B)$, *where* $B = \{x \in X \mid \|x\| \le 1\}$. *If there exists a subsequence* $\{(x_{n_k}, \alpha_{n_k})\}$ *which converges to* $(x, \alpha)$, *then* $\{(x_n, \alpha_n)\}$ *converges to* $(x, \alpha)$.

*Proof.* Since $\{(x_n, \alpha_n)\}$ is a fuzzy $\alpha$-Cauchy sequence, for each $t > 0$, there exists a positive integer $M$ such that $n, m \ge M$ implies $\alpha_n \wedge \alpha_m \le \alpha$ and $P_{(\alpha_n \wedge \alpha_m)^-}(x_n - x_m) < t$. And since $\|x_n - x_m\| =$

$P_{(\alpha_n \wedge \alpha_m)^-}(x_n - x_m)$ by Lemma 2.1, $\{x_n\}$ is a crisp Cauchy sequence in $X$. Thus $\{x_{n_k}\}$ is a crisp subsequence of $\{x_n\}$ in $X$. Since $\{x_{n_k}\}$ converges to $x$ by Lemma 2.2, $\{x_n\}$ converges to $x$. Since $\{(x_n, \alpha_n)\}$ is a fuzzy $\alpha$-Cauchy sequence, $\alpha_n \leq \alpha$ for sufficient large $n$. Therefore for each $t > 0$,there exists a positive integer $M$ such that $n \geq M$ implies $\alpha_n \leq \alpha$ and $P_{\alpha_n^-}(x_n - x) = \|x_n - x\| < t$ equivalently $\{(x_n, \alpha_n)\}$ converges to $(x, \alpha)$. □

THEOREM 2.4. ([1, p194]) *Let $W$ be a closed convex set in a uniformly convex Banach space $X$. Then there exists a "best approximation" projection $p$ from $X$ onto $W$, that is, for all $x \in X$, there is a unique point $y = p(x) \in W$ such that*

$$\|x - y\| = \inf\{\ \|x - z\| \mid z \in W\ \}.$$

THEOREM 2.5. *Let $X$ be a uniformly convex Banach space over $K$. Then $\rho = \chi_{B_{\|\cdot\|}}$ is a fuzzy seminorm on $X$, where $B_{\|\cdot\|} = \{x \in X \mid \|x\| \leq 1\}$. Let $W \subseteq X$ be a closed subspace and $q : X \to X/W$ the quotient map. Then $\rho_q = \chi_{B_{|||\cdot|||}}$, where for $x + W \in X/W$,*

$$\rho_q(x + W) = \vee\{\ \rho(x + y) \mid y \in W\ \},$$
$$|||x + W||| = \wedge\{\ \|x + y\| \mid y \in W\ \},$$
$$B_{|||\cdot|||} = \{\ x + W \mid |||x + W||| \leq 1\ \}.$$

*Proof.* Since $\rho_q(x + W) = 0$ or $1$ for all $x \in X$, $\rho_q(x + W) = 0$ if and only if for all $y \in W$, $\|x + y\| > 1$. To prove the theorem, it is sufficient to show that $|||x + W||| > 1$ if and only if for each $y \in W$, $\|x + y\| > 1$. Suppose for each $y \in W$, $\|x + y\| > 1$. Let $A = \{\|x+y\| \mid y \in W\}$. Then $\|x+y\| \in A$ implies $\|x+y\| > 1$. By the preceding theorem $\wedge_{y \in W}\|x + y\| = \|x + y'\|$ for some $y' \in W$. And so $\wedge_{y\in W}\|x+y\| \in A$. Therefore $|||x+W||| = \wedge_{y\in W}\|x+y\| > 1$. Conversely, suppose $|||x+W||| > 1$, i.e., $\wedge_{y\in W}\|x+y\| > 1$. Then for each $y \in W$, $\|x+y\| \geq \wedge_{y\in W}\|x+y\| > 1$. □

COROLLARY 2.6. *Let $X$ be a uniformly convex Banach space, $\rho = \chi_B$ and $W$ be a closed subspace of $X$, where $B = \{x \in X \mid \|x\| \leq 1\}$. Let $\{(x_n+W, \alpha_n)\}$ be a fuzzy $\alpha$-Cauchy sequence in the fuzzy normed linear space $(X/W, \rho_q)$. If there exists a subsequence $\{(x_{n_k}+W, \alpha_{n_k})\}$ which converges to $\{(x+W,\alpha)\}$. Then $\{(x_n+W,\alpha_n)\}$ converges to $\{(x+W,\alpha)\}$.*

THEOREM 2.7. ([7, Theorem 3.7]) *Let $(X, \|\cdot\|)$ be a Banach space. Then the fuzzy normed linear space $(X, \chi_B)$ is fuzzy complete, where $B = \{\ x \in X \mid \|x\| \leq 1\}$.*

COROLLARY 2.8. *Let $X$ be a Banach space and $\rho = \chi_B$, where $B = \{x \in X \mid \|x\| \leq 1\}$. If $W \subseteq X$ is a closed subspace of $X$, then $(W, \rho)$ is a fuzzy Banach space.*

COROLLARY 2.9. *Let $X$ be a uniformly convex Banach space, $\rho = \chi_B$, where $B = \{x \in X \mid \|x\| \leq 1\}$. If $W$ is a closed subspace of $X$. Then $(X/W, \rho_q)$ is a fuzzy Banach space.*

THEOREM 2.10. ([8, Theorem 3.11]) *Let $(X, \|\cdot\|)$ be a normed linear space and $\rho$ a fuzzy norm. If $\rho$ is lower semi continuous and has the bounded support, then following statements are equivalent.*

(1) *For some $\alpha \in (0,1)$, $(X,\rho)$ is fuzzy $\alpha$-complete.*
(2) *For some $\alpha \in (0,1)$, $(X, P_{\alpha^-})$ is complete.*
(3) *$(X,\rho)$ is fuzzy complete.*
(4) *$(X,\chi_B)$ is fuzzy complete.*
(5) *$(X, \|\cdot\|)$ is complete.*

COROLLARY 2.11. *Let $\rho$ be a fuzzy norm on a Banach space $X$. And let $W$ be a closed subspace of $X$. If $\rho_q$ is lower semi continuous and has the bounded support, then $(X/W, \rho_q)$ is a fuzzy Banach space, where $\rho_q(x+W) = \vee\{\rho(x+y) \mid y \in W\}$, $x+W \in X/W$.*

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