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EXTENSIONS OF FUZZY IDEALS IN NEAR–RINGS

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ABSTRACT. We characterize fuzzy ideals in near-rings and extensions of such ideals with the sup property.

1. Introduction

Fuzzy ideals of rings was introduced by W. Liu [10], and it has been studied by several authors [4, 7, 8]. The notion of fuzzy ideals and its properties were applied to various areas: semigroups [9, 12], distributive lattices [15], BCK-algebras [14], near-rings [1, 6]. S. Abou-Zaid [1] studied fuzzy subnear-rings and ideals, and S. D. Kim and H. S. Kim [6] investigated further fuzzy ideals of near-rings. The notion of L-fuzzy ideals was applied to semirings by J. Neggers, Y. B. Jun and H. S. Kim [13]; to BCK-algebras by Y. B. Jun, E. H. Roh and H. S. Kim [5].

In this paper we consider conditions when it is possible to construct an extension of a fuzzy left (resp. right) ideal μ of a subnear-ring S of a near-ring R to a fuzzy left (resp. right) ideal μ^e of R such that μ and μ^e have the same image.

Now, we review some definitions and results which will be helpful for the discussion.

A non-empty set R with two binary operations '$+$' and '$\cdot$' is called a *near-ring* ([3]) if

(1) $(R, +)$ is a group,

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(2) $(R, \cdot)$ is a semigroup,
(3) $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

We will use the word 'near-ring' to mean 'left near-ring'. We denote xy instead of $x \cdot y$.

Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in R$.

An *ideal* I of a near-ring R is a subset of R such that

(4) $(I, +)$ is a normal subgroup of $(R, +)$,
(5) $RI \subseteq I$,
(6) $(r+i)s - rs \in I$ for any $i \in I,\ r, s \in R$.

Note that I is called a *left ideal* of R if I satisfies (4) and (5), and I is called a *right ideal* of R if I satisfies (4) and (6).

Let R be a near-ring and μ be a fuzzy subset of R. We say μ a *fuzzy subnear-ring* of R if

(7) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,
(8) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

μ is called a *fuzzy ideal* of R if μ is a fuzzy subnear-ring of R and

(9) $\mu(x) = \mu(y + x - y)$,
(10) $\mu(xy) \geq \mu(y)$,
(11) $\mu((x+i)y - xy) \geq \mu(i)$ for any $x, y, i \in R$.

Note that μ is called a *fuzzy left ideal* of R if it satisfies (7), (9) and (10) and μ is called a *fuzzy right ideal* of R if it satisfies (7), (8), (9) and (11), (See [1]).

EXAMPLE 1.1.([6]) Let $R = \{a, b, c, d\}$ be a set with two binary operations as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Then we can easily see that $(R, +, \cdot)$ is a (left) near-ring. Define a fuzzy subset $\mu : R \to [0,1]$ by $\mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then μ is a fuzzy ideal of R.

LEMMA 1.2. ([6]) *If a fuzzy subset μ of a near-ring R satisfies the property* (7) *then*

(i) $\mu(0_R) \geq \mu(x)$,
(ii) $\mu(-x) = \mu(x)$, *for all* $x \in R$.

Throughout the paper R will denote a left near-ring. Let μ be a fuzzy subset of a near-ring R. For $\alpha \in [0,1]$, the set $\mu_\alpha := \{x \in R | \mu(x) \geq \alpha\}$ is called a *level subset* of μ.

THEOREM 1.3. ([1]) *Let μ be a fuzzy subset of a near-ring R. Then the level subset μ_α is a subnear-ring (or ideal) of R for all $\alpha \in [0,1], \alpha \leq \mu(0)$ if and only if μ is a fuzzy subnear-ring (or ideal), respectively.*

Let X be a non-empty (usual) set. $\mathcal{F}(X)$ will denote the set of all fuzzy subsets in X. If $\mu, \nu \in \mathcal{F}(X)$, then $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$, and $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$. It is easily seen that $\mathcal{F}(X) = (\mathcal{F}(X), \subseteq, \wedge, \vee)$ is a completely distributive lattice.

We note that the intersection of all left (resp. right and two sided) ideals of a near-ring R is also a left (resp. right and two sided) ideal of R. Let Λ be a totally ordered set and let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of left (resp. right) ideals of R such that for all $\alpha, \beta \in \Lambda$, $\beta > \alpha$ if and only if $A_\beta \subset A_\alpha$. Then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is a left (resp. right) ideal of R.

DEFINITION 1.4. A fuzzy subset μ in a set X has the *sup property* if for any subset T of X, there exists $t_0 \in T$ such that

$$\mu(t_0) = \sup_{t \in T} \mu(t).$$

DEFINITION 1.5.([11]) Let X be a non-empty set. By an *extension* of $\mu \in \mathcal{F}(X)$ to a fuzzy subset ν in a set R containing X, we mean a fuzzy subset ν in R such that $\nu = \mu$ in X.

LEMMA 1.6. ([11]) *Let X be a non-empty subset of a set R and let $\mu \in \mathcal{F}(X)$ be such that μ has the sup property. If $\mathcal{B} = \{B_\alpha | \alpha \in \mathrm{Im}(\mu)\}$ is a collection of subsets of R such that*

(i) $\bigcup_{\alpha \in \mathrm{Im}(\mu)} B_\alpha = R$,
(ii) *$\beta > \alpha$ if and only if $B_\beta \subset B_\alpha$ for all $\alpha, \beta \in \mathrm{Im}(\mu)$,*
(iii) *$\mu_\alpha \cap B_\beta = \mu_\beta$ for all $\alpha, \beta \in \mathrm{Im}(\mu)$, $\beta \geq \alpha$,*

then μ has a unique extension $\mu^e \in \mathcal{F}(R)$ such that $(\mu^e)_\alpha = B_\alpha$ for all $\alpha \in \mathrm{Im}(\mu)$ and $\mathrm{Im}(\mu^e) = \mathrm{Im}(\mu)$.

2. Extensions of fuzzy ideals

In this section we characterize fuzzy left (resp. right) ideals of R, and obtain some properties of extensions when the ideals have the sup property.

THEOREM 2.1. *If μ is a fuzzy left (resp. right) ideal of R, then $\mu(x) = \sup\{\alpha \in L | x \in \mu_\alpha\}$ for all $x \in R$.*

Proof. Let $\beta = \sup\{\alpha \in L | x \in \mu_\alpha\}$ and $\varepsilon > 0$ be given. Then

$$\beta - \varepsilon < \sup\{\alpha \in L | x \in \mu_\alpha\},$$

whence $\beta - \varepsilon < \alpha$ for some $\alpha \in L$ such that $x \in \mu_\alpha$. Since $\mu(x) \geq \alpha$, it follows that $\beta - \varepsilon < \mu(x)$, so that $\beta \leq \mu(x)$ because ε is arbitrary. We now show that $\mu(x) \leq \beta$. To do this, assume that $\gamma = \mu(x)$. Then $x \in \mu_\gamma$ and so $\gamma \in \{\alpha \in L | x \in \mu_\alpha\}$. Hence $\gamma \leq \sup\{\alpha \in L | x \in \mu_\alpha\}$, whence $\mu(x) \leq \beta$. Therefore $\mu(x) = \beta$, as desired. □

Now we consider the converse of Theorem 2.1. Let Λ be a non-empty subset of $[0, 1]$. Without loss of generality we use Λ as an index set in the following results.

THEOREM 2.2. *Let $\{A_\alpha | \alpha \in \Lambda\}$ be a collection of left (resp. right) ideals of R such that*

(i) $R = \bigcup_{\alpha \in \Lambda} A_\alpha$,

(ii) *$\beta > \alpha$ if and only if $A_\beta \subset A_\alpha$ for all $\alpha, \beta \in \Lambda$. Define $\mu \in \mathcal{F}(R)$ by, for all $x \in R$,*

$$\mu(x) := \sup\{\alpha \in \Lambda | x \in A_\alpha\}.$$

Then μ is a fuzzy left (resp. right) ideal of R.

Proof. For any $\beta \in [0, 1]$, we consider the following two cases:

$$(1)\ \beta = \sup\{\alpha \in \Lambda | \alpha < \beta\},$$

$$(2)\ \beta \neq \sup\{\alpha \in \Lambda | \alpha < \beta\}.$$

For the case (1), we know that

$$x \in \mu_\beta \Leftrightarrow x \in A_\alpha \text{ for all } \alpha < \beta \Leftrightarrow x \in \bigcap_{\alpha < \beta} A_\alpha,$$

whence $\mu_\beta = \bigcap_{\alpha<\beta} A_\alpha$, which is a left (resp. right) ideal of R. Case (2) implies that there exists $\varepsilon > 0$ such that $(\beta-\varepsilon, \beta) \cap \Lambda = \emptyset$. We claim that $\mu_\beta = \bigcup_{\alpha\geq\beta} A_\alpha$. If $x \in \bigcup_{\alpha\geq\beta} A_\alpha$, then $x \in A_\alpha$ for some $\alpha \geq \beta$. It follows that $\mu(x) \geq \alpha \geq \beta$, so that $x \in \mu_\beta$. Conversely if $x \notin \bigcup_{\alpha\geq\beta} A_\alpha$, then $x \notin A_\alpha$ for all $\alpha \geq \beta$, which implies that $x \notin A_\alpha$ for all $\alpha > \beta-\varepsilon$, that is, if $x \in A_\alpha$ then $\alpha \leq \beta - \varepsilon$. Thus $\mu(x) \leq \beta - \varepsilon$, and so $x \notin \mu_\beta$. Therefore $\mu_\beta = \bigcup_{\alpha\geq\beta} A_\alpha$, which is a left (resp. right) ideal of R. Using Lemma 1.3, we see that μ is a fuzzy left (resp. right) ideal of R. □

Let S be a subnear-ring of R. For a left (right, two sided, resp.) ideal A generated by S, let A^e denote the left (right, two sided, resp.) ideal of R generated by A.

THEOREM 2.3. *Let S be a subnear-ring of R and let μ be a fuzzy ideal of S such that μ has the sup property. If $\bigcup_{\alpha\in\mathrm{Im}(\mu)} (\mu_\alpha)^e = R$ and $\mu_\alpha \cap (\mu_\beta)^e = \mu_\beta$ for all $\alpha, \beta \in \mathrm{Im}(\mu)$ with $\beta \geq \alpha$, then μ has a unique extension to a fuzzy left (resp. right) ideal μ^e of R such that $(\mu^e)_\alpha = (\mu_\alpha)^e$ for all $\alpha \in \mathrm{Im}(\mu)$ and $\mathrm{Im}(\mu^e) = \mathrm{Im}(\mu)$.*

Proof. Since $\beta > \alpha$ if and only if $\mu_\beta \subset \mu_\alpha$ for all $\alpha, \beta \in \mathrm{Im}(\mu)$, the condition $\mu_\alpha \cap (\mu_\beta)^e = \mu_\beta$ implies that $\beta > \alpha$ if and only if $(\mu_\beta)^e \subset (\mu_\alpha)^e$. If we let $B_\alpha = (\mu_\alpha)^e$, then by Lemma 1.6 we see that μ has a unique extension to $\mu^e \in \mathcal{F}(R)$ such that

$$(\mu^e)_\alpha = (\mu_\alpha)^e,$$
$$\mathrm{Im}(\mu^e) = \mathrm{Im}(\mu)$$

for all $\alpha \in \mathrm{Im}(\mu)$. Noticing that $(\mu^e)_\alpha = (\mu_\alpha)^e$ is a left (resp. right) ideal of R, and using Lemma 1.3, we conclude that μ^e is a fuzzy left (resp. right) ideal of R. This completes the proof. □

REFERENCES

1. S. Abou-Zaid, *On fuzzy subnear-rings and ideals*, Fuzzy Sets & Sys. **44** (1991), 139-146.
2. P. Bhattacharya and N. P. Mukherjee, *Fuzzy relations and fuzzy groups*, Inform. Sci. **36** (1985), 267-282.
3. J. R. Clay, *Nearrings; Geneses and Applications*, Oxford, New York (1992).
4. Y. N. Dixit, R. Kumar and N. Ajal, *On fuzzy rings*, Fuzzy Sets & Sys. **49** (1992), 205-213.
5. Y. B. Jun, E. H. Roh and H. S. Kim, *Extensions of fuzzy ideals in BCK algebras*, submitted.
6. S. D. Kim and H. S. Kim, *On fuzzy ideals of near-rings*, Bull. Korean Math. Soc. **33** (1996), 593-601.
7. R. Kumar, *Fuzzy irreducible ideals in rings*, Fuzzy Sets & Sys. **42** (1991), 369-379.
8. R. Kumar, *Certain fuzzzy ideals of rings redefined*, Fuzzy Sets & Sys. **46** (1992), 251-260..
9. N. Kuroki, *On fuzzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets & Sys. **5** (1981), 203-205.
10. W. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets & Sys. **8** (1982), 133-139.
11. D. S. Malik and J. N. Mordeson, *Extensions of fuzzy subrings and fuzzy deductive systems*, Fuzzy Sets & Sys. **45** (1992), 245-251.
12. R. G. Mclean and H. Kummer, *Fuzzy ideals in semigroups*, Fuzzy Sets & Sys. **48** (1992), 137-140.
13. J. Neggers, Y. B. Jun and H. S. Kim, *Extensions of L-fuzzy ideals in semirings*, submitted.
14. X. Ougen, *Fuzzy BCK-algebras*, Math. Japonica **36** (1991), 935-942.
15. Yuan Bo and Wu Wangming, *Fuzzy ideals on a distributive lattices*, Fuzzt sets & Sys. **35** (1990), 231-240.

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