산업기술연구(강원대학교 산업기술연구소 논문집), 제17집, 1997
Research Report. The Institute of Industrial Technology, Kangwon Nat'l Univ., Korea, Vol. 17, 1997.

앞쪽으로 진행되며 이동하는 파를 해로 갖는 편미분 방정식이 움직이는 근원을 가질 경우의 해에 대한 연구

한 춘 호*

A note on wavefront-type travelling wave solutions with moving sources in several systems.

Han, Chun-Ho*

ABSTRACT

이 논문에서는 이동하는 파를 해로 갖는 편미분 방정식의 해를 찾는 방법에 대하여 연구하였다. 특히 이 해가 움직이는 근원을 가질 경우에 앞쪽으로 이동하는 해가 존재하는지의 여부를 파악하는데 어떻게 증명할 것인지를 보였다. 앞쪽으로 이동하는 해를 갖는 경우는 반응-확산 방정식에서 그 해를 찾아 볼 수 있으며, 움직이는 근원을 가진 경우에 앞쪽으로 진행되는 해를 갖는 경우는 화학 공정의 모델등에서 그 해를 찾아 볼 수 있는데 이 문제를 중심으로 어떻게 해의 존재 여부를 증명할 수 있는지를 보였다.

1. Introduction

Recently, many applied mathematicians are interested in the mathematical models of physics, chemical reactions, biological systems, fluid mechanics, acoustics, elasticity and electromagnetic theory which have the travelling wave solutions. Consider the basic partial differential equation

$$v_t - Av_{xx} = 0,\ x \in R,\ t > 0$$
$$v(x,0) = v_0(x),\ x \in R \quad \text{.........(1.1)}$$

which has the well known solution

$$v(x,t) = \frac{1}{\sqrt{4\pi At}} \int_R v_0(z)\, e^{-\frac{(x-z)^2}{4At}}\, dz \quad \text{.........(1.2)}$$

The well known advection(partial differential) equation with initial value problem having the travelling wave solution is

$$v_t + cv_x = 0,\ x \in R,\ t > 0$$
$$v(x,0) = v_0(x),\ x \in R \quad \text{.........(1.3)}$$

where c is a positive constant. This has the solution $v(x,t) = f(x-ct)$ which can be easily checked. So the global solution of the above equation is

$$v(x,t) = v_0(x,t),\ x \in R,\ t > 0.$$

Graphically the solution is the initial signal $u_0(t)$ moving to the right by the amount ct without any distortion. The general type of signalling problem is

* 강원대학교 공과대학 토목공학과 부교수
* 이 논문은 강원대학교 기성회 학술 연구 지원으로 작성된 논문임.

$$u_t + F(x, t, v, v_x, v_{xx}) = 0, x>0, t>0$$
$$v(0, t) = v_0(x), x>0$$
$$v(0, t) = v_1(t), t>0$$
$$v_x(0, t) = v_2(t), t>0 \quad (1.4)$$

One of the important problem of nonlinear partial differential equations is wave propagation. A wave means a regonizable signal which is transferred from one part to another part in time t with the same speed. One of the simplest form of a mathematical wave is a function of the form

$$v(x, t) = f(x+ct) + g(x-ct) \quad (1.5)$$

where f(x+ct) and g(x-ct) represent a left travelling wave of speed c and a right travelling wave of speed c, respectively. Usually, boundary conditions of the form

$$v(-\infty, t) = constant$$
$$v(\infty, t) = constant$$

are imposed. A wavefront-type solution to a partial differential equation is a solution of the form $v(x, t) = f(x-ct)$ with the above condition where the constants must not be the same.

2. Reaction-Diffusion equation.

Many problems have the both diffusion and reaction mechanisms,

$$\overline{v}_{\overline{t}} - D\,\overline{v}_{\overline{xx}} = f(\overline{v})$$
$$f(\overline{v}) = r\overline{v}(1 - \frac{\overline{v}}{k}) \quad (2.1)$$

which is the model of the diffusion of a species when the reaction or growth term is given by the logistic law. Here *D*, *r* and *k* represent the diffusive constant, the growth rate and carrying capacity, respectively. Changing to the dimensionless quantities which is called scaling

$$t = r\overline{t}, \quad x = \frac{\overline{x}}{\sqrt{\frac{D}{r}}}, \quad v = \frac{u}{k} \quad (2.2).$$

Then the above barred equation becomes

$$v_t - v_{xx} = v(1-v) \quad (2.3)$$

The solution of this equation is $v(x, t) = V(z)$, where $z=x-ct$ (c is a positive constant). The function $U(z)$ goes to some constants when $z \to \pm\infty$. Then the equation becomes

$$-cV' - V'' = V(1-V), \quad -\infty < z < \infty,$$
$$V' = \frac{dV}{dz} \quad (2.4)$$

Since this problem is not linear, let's do linearize the system to get the characteristic of the equilibrium points of linearized system. So the autonomous system is

$$V' = Z$$
$$Z' = -V(1-V) - cZ \quad (2.5)$$

So the Jacobi matrix of the linearized system is

$$J = \begin{bmatrix} 0 & 1 \\ 2V-1 & -c \end{bmatrix},$$

and the eigenvalues of the linearized system are

$$\lambda_{\pm} = \frac{-c \pm \sqrt{c^2+4}}{2}$$

at the point (1,0), and

$$\lambda_{\pm} = \frac{-c \pm \sqrt{c^2-4}}{2}$$

at the point (0,0). They are real and of the opposite sign at the point (1,0), so (1,0) has the unstable saddle point. Also they are stable node if $c^2 \geq 4$, *i.e.* $c \geq 2$ at the point (0,0).

Now, discuss about travelling wave solutions of the system. The parameter z must tend to $\pm\infty$ as the path enters or exits a critical point.

So the separatrix connecting two equilibrium points are as follows;

$$V \to 1, \; as \; z \to -\infty$$
$$V \to 0, \; as \; z \to \infty.$$

And the right sides of the linearized system goes to 0 at the two critical points. This means

$$V' \to 0, \; as \; |z| \to \infty.$$

Thus the separatrix connecting two critical points is monotonic decreasing when $c \ge 2$.

3. Wavefront-type equation.

$$v_t + vv_x + q\mu_x = f(x-ct) \quad \text{.......}(3.1)$$
$$\mu_t = r(v, \mu), x \in R \quad \text{.......}(3.2)$$

where time variable t represents Lagrangian dimensionless space, and $v = v(x, t)$ and $\mu = \mu(x, t)$ represent a dimensionless temperature and chemical progress variable of a one-dimensional chemically reacting fluid undergoing a model reversible chemical reaction. Much of the work shows that for irreversible kinetics where the reaction occurs. The chemical reaction rate

$$r(v, \mu) = k_{f(v)}(1 - \mu - \mu e^{-\frac{a}{v}}) \quad \text{.......}(3.3)$$

where $k_{f(v)}$ is the forward, temperature dependent, rate constant.
The solution of (2.1) is in C^1 of the form

$$v = v(z), \mu = \mu(z)$$
$$where \; z = x - ct \; ; \; z \in R \quad \text{.......}(3.4)$$
$$v(\pm\infty) = v_{\pm}, \quad \mu(\pm\infty) = \mu_{\pm} \quad \text{.......}(3.5)$$
$$v', \mu' \to 0 \quad as \quad |z| \to \infty \quad \text{.......}(3.6)$$

Here the wave speed c is not known yet, but we can determine c when the wavefront-type solution exists. Substitute (2.4)-(2.6) into (2.1) and (2.2), then we get

$$(v-c)(v-c)' + q\mu' = f(z) \quad \text{.......}(3.7)$$
$$\mu' = -\frac{1}{c} r(v, \mu) \quad \text{.......}(3.8)$$

Integrate f directly, this can be done because f is continuous, positive and integrable on R. Then we get

$$\frac{1}{2}(v_- - c)^2 + q\mu$$
$$= \frac{1}{2}(v_+ - c)^2 + q\mu_+ - \int_z^{\infty} f(v)dv \quad \text{.......}(3.9)$$

We can determine the constant of integration from (2.7). Define

$$||f|| = \int_R f(s)ds$$

Then the equation (3.9) must be positive to exist the wave-front type travelling wave solution.

Corollary 3.1]
A necessary condition for a wavefront-type travelling wave solution to exist is

$$\frac{1}{2}(v_- - c)^2 + q\mu_-$$
$$= \frac{1}{2}(v_+ + c)^2 + q\mu_+ - ||f|| > 0 \quad \text{.......}(3.10)$$

Solve the equation (2.10) for c, then we get the wave speed c

Corollary 3.2]
A necessary condition for a wavefront-type travelling wave solution to exist is that the wavespeed is given by

$$c = \frac{\frac{1}{2}(v_+ + c)^2 + q\mu_+ - (\frac{1}{2}(v_- - c)^2 + q\mu_-) - ||f||}{v_+ - v_-} > 0$$
$$\text{.......}(3.11)$$

If c<0 (when $||f||$ is so large), the condition must be failed. This means $||f||$ i.e. the toatl energy of the heat source is adquately small. This imply that $r(v_\pm, \mu_\pm)=0$, i.e. the state at $z=\pm\infty$ must lie on the equilibrium curve and

$$\mu=\frac{1}{1+e^{-\frac{a}{v}}} \quad ....(3.12)$$

Equation (2.12) graphs as a family of the graph of concave down parabolic shapes which miximum occur at $v=c$.

4. Existence of a travelling wave solution.

First we want to show the existence of a wavefront-type travelling wave solution of (3.1)-(3.3) of the case $v_+ < v_- < c$; this will be a solution of a weak supersonic wave. Solving the problem (3.10) for v, we get

$$v=c-\sqrt{(v_+-c_+)^2+2\int_z^\infty f(z)dz} ......(4.1)$$

Since f is strictly decreasing and substitute (3.1) into (2.8) to get

$$\mu'=-\frac{r}{c}(c-\sqrt{(v-c)^2-2\int_z^\infty f(z)dz}, \mu)(4.2)$$

Since

$$v_+ < c-\sqrt{(v-c)^2-2\int_z^\infty f(z)dz}=v ...(4.3),$$

we get

$$r(c-\sqrt{(v-c)^2-2\int_z^\infty f(z)dz}, \mu_-) < r(v_+, \mu_+)=0 \quad ....(4.4)$$

So we can get the next Corollary from (4.2).

Corollary 4.1]

$$\mu'=-\frac{r}{c}(c-\sqrt{(v-c)^2-2\int_z^\infty f(z)dz}, \mu)>0 \quad ....(4.5)$$

Now we want to determine the properties of the null cline, i.e. the locus where $\mu'=0$; this is the curve $\mu=\overline{\mu}(z)$ defined by $r(v(\mu,z),\mu)=0$. Along the line $r=0$

$$\frac{d\mu}{dz}=\frac{d\mu}{dv}\frac{dv}{dz}=\frac{d\mu}{dv}\frac{q\frac{d\mu}{dz}-f(z)}{\sqrt{(v_+-c)^2-2\int_z^\infty f(z)dz}} \quad ....(4.6).$$

Let

$$Q=-\frac{d\mu}{dv}\frac{1}{\sqrt{(v_+-c)^2-2\int_z^\infty f(z)dz}}>0.$$

Then (4.6) becomes

$$(1+qQ)\frac{d\mu}{dz}=Qf(z) \quad ....(4.7)$$

since $\frac{d\mu}{du}<0$ from (2.12). Consequently the function $\mu=\overline{\mu}(z)$ defined by

$$-\frac{r}{c}(c-\sqrt{(v-c)^2-2\int_z^\infty f(z)dz}, \mu)=0 \quad ....(4.8)$$

is increasing on the interval $-\infty < z < \infty$ and $\overline{\mu}\to\mu_\pm$ as $z\to\pm\infty$. Therefore we can get the next Theorem.

Theorem 4.1]
Suppose f is a positive continuous integrable function on R and the rate function is given by (2.4). Consider the system (2.1) and (2.2) of the form with the boundary conditions (2.5) and (2.6). Then nexts hold when $v_+>0$ and $q>0$ are given, and $c>0$ and $v_->0$ are chosen.

1) $v_+ < v_- < c$

2) $\frac{1}{2}(v-c)^2+q\mu$ has the solution

(v_-, μ_-), where $\mu_- = \frac{1}{1+e^{-\frac{a}{v_-}}}$

3) travelling wave solutions exist.

5. Conclusion

Usually the solutions of partial differential equation can't be found easily. In this paper we investigate the solutions of the partial differential equations which has the travelling wave solutions. Especially we try to find the solutions which have the wavefront-type. Usually the type of the solutions appears in the reaction-diffusion model. So first we try to find some solutions of reaction-diffusion equation which has the travelling wave solution. Next we try to find wavefront-type solutions with moving sources in the medium. So we pick one problem which can be arisen from the chemical progress. This is the problem of an one dimensional chemically reacting fluid under-going a model reversible chemical reaction. Here we try to show how can we prove the existence of such travelling wave solution.

6. References.

1. Rodolfo R Rosales & Andrew Majda, : *Weakly nonlinear detonation waves*, SIAM J. Appl. Math., Vol 43, No. 5, pp. 1086-1118, 1983

2. Wildon Fickett : *Detonation in miniature*, Am. J. Phys., Vol. 47, No.12, pp. 1050-1059, 1979

3. Glenn Ledder : *Weakly nonlinear asymptotic models and analogs of detonation processes*, Int. J. Eng. Sci., Vol. 30, No.12, pp. 1759-1772, 1992

4. Glenn Ledder & J. Logan : *A signalling problem for near-equilibrium flows in the Fickett - Majda model of combustion*, IMA J. App. Math., Vol. 47, pp.229-246, 1991

5. J. D. Murray : *Mathematical biology, Springer-Verlag*, New York, 1989

6. C. C. Lin : *Mathematical applied to deterministic problems in the natural science*, SIAM, 1988

7. Paul Waltman : *A second course in elementary differential equation*, Academic Press, 1986

8. J. D. Logan : *Applied mathematics*, A Wiley-Interscience Pub., 1961

9. William F. Ames : *Numerical methods for partial differential equations*, Academic Press, 1992

10. Ronald B. Guenther & John W. Lee : *Partial differential equations of mathematical physics and integral equations*, Prentice Hall, 1988