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비선형 현수교 방정식의 주기함수로 나타나는 해에 대한 연구

한 춘 호* · 심 도 식**

A note on the periodic solutions of the nonlinear suspension bridge equation

Han, Chun-Ho* · Shim, Do-Sik**

ABSTRACT

이 논문에서는 비선형 빔방정식을 이용할 수 있는 현수교방정식의 존재하는 해의 개수를 조사하였다. 외부에서 주어지는 함수가 주기함수일 경우에 나타나는 여러 가지 성질들을 조사하였으며 주어진 항들의 계수가 상수인 경우 어떤 범위에서 몇 개의 해가 존재할 수 있는지를 조사하였다. Leray-Schauder degree를 이용하여 존재할 수 있는 해의 개수를 판별하는 근거로 삼았다. 특히 일정한 항의 계수가 변수를 포함하는 경우에 나타날 수 있는 변화에 대하여 조사하였다.

1. Introduction

Some of the suspension bridge equation can be represented by a non-linearly supported vibrating beam. Consider an one-dimensional beam of length L suspended by cables. When the cables are stretched, there is a restoring force which is assumed to be proportional to the amount of the stretching. But when the beam moves in the opposite direction, then there is no restoring force exerted on it. If $u(x,t)$ denotes the displacement in the downward direction at position x and time t, then a simplified model is given by the equations

$$\begin{aligned} &u_{tt}+A_1 u_{xxxx}+A_2 u = W(x)+\varepsilon f(x,t) \\ &u(x,0)=u(L,t)=0 \\ &u_{xx}(0,t)=u_{xx}(L,t)=0 \qquad \text{......................(1.1)} \end{aligned}$$

in which $W(x)$ is the weight per unit length at x, and $f(x,t)$ is an externally imposed periodic function. These equations can be considered under the assumption that

$$W(x)=W_0 \sin\frac{\pi x}{L},$$

which allowed the partial differential equation. It's well known that, if A_2 is large enough, then large numbers of highly oscillatory solutions could exist.

2. Periodic Solutions

We begin this section investigating periodic solutions of the problem (1.1) under the more realistic assumption that the weight per unit length is constant, *i.e.* $W(x)=W_0, 0<x<L$. By obvious change of variables, problem (1.1) can be reduced to

* 강원대학교 공과대학 토목공학과 부교수
** 강원대학교 공과대학 토목공학과 시간강사, 공학박사

$$u_{tt}+u_{xxxx}+bu^{+}=1+\varepsilon f(x,t)$$
$$u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=0 \quad in \quad (-\frac{\pi}{2},\frac{\pi}{2})\times R \quad ..(2.1)$$

Without loss of generality, we can assume $f(x,t)$ is even in x and t, and periodic with period π, and we shall look for the π-periodic solutions of (1.2). Part of our analysis concerns the steady-state case of (1.2). A positive force c produces a steady-state deflection $v(x)$ satisfying

$$v''''+bv^{+}=c, \quad in \quad (-\frac{\pi}{2},\frac{\pi}{2})$$
$$v(\pm\frac{\pi}{2})=v''(\pm\frac{\pi}{2})=0 \quad \text{..........................}(2.2)$$

Physical intuition suggests that v is positive, as the beam deflects under the load, and one would also expect that the externally imposed π-periodic force $\varepsilon f(x,t)$ produces small oscillations of the order of magnitude ε around the steady-state solution. It's easy to demonstrate that for certain ranges of b additional highly oscillatory π-periodic solutions, which change sign, also exist.

Let L be the differential operator

$$Lu=u_{tt}+u_{xxxx} \quad \text{......................................}(2.3)$$

The eigenvalue problem for $u=u(x,t)$, which is even in x and t and π-periodic in t, is

$$Lu=\lambda u \text{ in } (-\frac{\pi}{2},\frac{\pi}{2})\times R,$$
$$u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=0$$
$$u(x,t)=u(-x,t)=u(x,-t)=u(x,t+\pi) \quad \text{..............................}(2.4)$$

which has infinitely many eigenvalues λ_{mn} and corresponding eigenfunctions $\phi_{mn}(m,n\geq 0)$ given by

$$\lambda_{mn}=(2n+1)^2-4m^2$$
$$\phi_{mn}=\cos 2mt\cdot\cos(2n+1)x$$
$$(m,n=0,1,2,3,\cdots\cdots)$$

Investigating all eigenvalues to find out the eigenvalues in the interval *(-19,45)*, we can find the eigenvalues easily

$$\lambda_{20}=-15<\lambda_{10}=-3<\lambda_{00}=1<\lambda_{41}=17.$$

The normalized eigenfunctions are denoted by

$$\theta_{mn}=\frac{\phi_{mn}}{||\phi_{mn}||} \quad \text{where} \quad ||\phi_{mn}||=\frac{\pi}{2} \quad \text{for} \quad m>0,$$
$$||\phi_{0n}||=\frac{\pi}{\sqrt{2}}.$$

Let Q be the square in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, and H be the Hilbert space defined by

$$H=\{\, u\in L_2(\Omega): u \text{ is even in } x \text{ and } t\}.$$

The set of functions $\{\theta_{mn}\}$ is an orthonormal base in H.

The weak solution of the problem

$$u_{tt}+u_{xxxx}=f(u,x,t) \text{ in } (-\frac{\pi}{2},\frac{\pi}{2})\times R,$$
$$u(\pm\frac{\pi}{2},t)=u_{xx}(\pm\frac{\pi}{2},t)=0$$

u is even in x and t and π-periodic in t is of the form

$$u=\sum c_{mn}\theta_{mn}$$
$$\text{with } Lu=\sum\lambda_{mn}c_{mn}\theta_{mn}\in H,$$

i.e. with $u=\sum c_{mn}^2\theta_{mn}^2<\infty$ which implies $u\in H$. And f will be such that $u\in H$ implies $f(u,x,t)\in H$.

Lemma 2.1]
For *-1<b<15*, the problem

$$Lu+bu^{+}=0 \text{ in } H \quad \text{.............................(2.5)}$$

has only trivial solution $u=0$.

Proof]

The space $H_1 = span\{\cos x \cos 2mt: m \ge 0\}$ is invariant under L and under the map $u \to bu^{+}$. The spectrum σ_1 of L restricted to H_1 contains $\lambda_{10}=-3$ and does not contain any other point in the interval *(-15,1)*. The spectrum σ_2 of L restricted to $H_2 = H_1^{\perp}$ does not intersect the interval *(-15,1)*. So we conclude that any solution of $Lu+bu^{+}=0$ belongs to H_1, i.e. it is of the form $y(t)\cos x$ where y satisfies

$$y''+by^{+}+y=0.$$

Any non trivial solution periodic solution of this equation is periodic with period

$$\frac{\pi}{\sqrt{b+1}}+\frac{\pi}{1} \neq \pi.$$

So there is no non trivial solution of $Lu+bu^{+}=0$ in H. ■

Lemma 2.2]

Let $h \in H$ with $\|h\|=1$ and $a>0$ be given. There exists $R_0 = R_0(h,a) > 0$ such that for all b with $-1+a \le b \le 15-a$ and for all $\varepsilon \in [-1,1]$ the solutions u of $Lu+bu^{+}=1+\varepsilon h$ in H satisfy $\|u\| < R_0$.

Proof]

Suppose $\|u\| < R_0$ doesn't hold. Then there exists sequence $\{b_n, \varepsilon_n, u_n\}$ with $b_n \in [a-1, 15-a]$, $|\varepsilon_n| \le n$, $\|u_n\| \to \infty$ such that

$$u_n = L^{-1}(1-b_n u_n^{+}+\varepsilon_n h).$$

■

The functions $v_n = \frac{v_n}{\|v_n\|}$ satisfy the equation

$$v_n = L^{-1}\left(\frac{1}{\|u_n\|} - b_n v_n^{+} + \frac{\varepsilon_n}{\|u_n\|} h\right).$$

Since L^{-1} is a compact operator, we may assume that $v_n \to v_0$ and $b_n \to b_0 \in (-1, 15)$. Since $\|v_n\|=1$, it follows that $\|v_0\|=1$ and

$$v_0 = L^{-1}(-b_0 v_0^{+})$$
$$Lv_0 + b_0 v_0^{+} = 0 \text{ in } H.$$

This contradicts to the result in the previous Lemma. ■

Lemma 2.3]

Under the same assumptions and with the notations of Lemma 2.2]

$$\deg_{LS}(u - L^{-1}(1-bu^{+}+\varepsilon h), B_R, 0) = 1$$

for all $R \ge R_0$, where $\deg_{LS}$ denotes the Leray-Schauder degree.

Proof]

If b=0, we have

$$\deg_{LS}(u - L^{-1}(1+\varepsilon h), B_R, 0) = 1$$

since the map is simply a transition of the identity and since $\|L^{-1}(1+\varepsilon h)\| < R_0$ by the previous Lemma. ■

Lemma 2.4]

The Green's function for the boundary value problem

$$v''''+bv^{+}=c, \ in \ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$v\left(\pm\frac{\pi}{2}\right) = v''\left(\pm\frac{\pi}{2}\right) = 0 \quad \text{..........................(2.6)}$$

is nonnegative if and only if $-1 < b \le c_0 = \frac{4k^4}{\pi^4}$, where k is the smallest positive zero of the function $\tan x - \tanh x$. ■

It's easy to get the values $x = 3.9266$ and $c_0 = 9.7617$.

Lemma 2.5]
For all $b > -1$, the unique solution v of

$$v'''' + bv^+ = c \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2})$$
$$v(\pm\frac{\pi}{2}) = v''(\pm\frac{\pi}{2}) = 0$$

is positive *in* $(-\frac{\pi}{2}, \frac{\pi}{2})$ and even in x and satisfies $v'(-\frac{\pi}{2}) > 0$ and $v'(\frac{\pi}{2}) < 0$

Proof]
The eigenvalues of $Mv = \lambda v$, where $M = D^4$ with the boundary conditions, are all greater or equal to 1. Hence, for any $c < 1$, $\|(M-c)^{-1}\| = \frac{1}{1-c}$. So $Mv = f(v, x, t)$ with $c \le f_v \le 1 - \varepsilon$ has a unique solution, since solutions v are characterized by

$$v = (M - c)^{-1}[f(v, x, t) - cv]$$

where the right hand side is Lipschitz continuous with a Lipschitz constant

$$c \le \frac{1 - \varepsilon - c}{1 - c} < 1. \quad ■$$

Lemma 2.6]
Let K be a compact set in $L_2 = L_2(\Omega)$, and let $\phi \in L_2$ be positive almost everywhere. Then there exists a modulus of continuity δ depending only on K and ϕ such that]

$$\|(\eta|\psi| - \phi)^+\| \le \eta\delta(\eta)$$
$$\text{for } \eta > 0 \text{ and } \psi \in K.$$

Proof]
The function $f_n: K \to R$ given by

$$f_n(\psi) = \|(\eta|\psi| - \phi)^+\|$$
$$(\psi \in K: n = 1, 2, 3, \cdots\cdots)$$

is continuous and the sequence $\{f_n\}$ is decreasing such that $f_n(\psi)$ converges pointwise for every $\psi \in K$. This convergence is uniform from the Dini's theorem. Define function $\delta(\eta)$ by

$$\delta(\eta) = \max \left\| (|\psi| \frac{-\phi}{\eta})^+ \right\| \text{ for } \psi \in K.$$

Then $\delta(\eta)$ is increasing in η; furthermore $\delta(\frac{1}{n}) \to 0$. So δ is a modulus of continuity. ■

Lemma 2.7]
Assume that $3 < b < 15$. Then there exist $\gamma > 0$, $\varepsilon_0 > 0$ such that

$$\deg_{LS}(u - L^{-1}(1 - bu^+ + \varepsilon h), B_\gamma(y), 0) = -1$$

for $|\varepsilon| < \varepsilon_0$, where v is the unique solution of (1.3)

3. Main Results

So far we proved some properties that can be occurred when we invesgate how many solutions can be found. The next two theorems follow from these Lemmas.

Theorem 3.1]
Let $h \in H$ with $\|h\| = 1$ and $3 < b < 15$. Then there exists $\varepsilon_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$ the equation

$$Lu + bu^+ = 1 + \varepsilon h \text{ in } H \quad \text{..........................(3.1)}$$

has at least two solutions.

Proof]
The equation (3.1) can be written in the form

$$Tu = u - L^{-1}(1 - bu^+ + \varepsilon h) = 0$$

Let B_a be a large ball of radius a. Then the Leray-Schauder degree of B_R for large $R > R_0$ is +1 by Lemma 2.6]. From Lemma 2.7], the degree on the ball $B_\gamma(v)$ is -1. So

$$d_{LS}(Tu, B_R \cap B_\gamma(v)^C, 0) = 2.$$

So $Lu + bu^+ = 1 + \varepsilon h$ has the solution one is in B_R, and one is in $B_R \cap B_\gamma(v)^C$. So we can reach to the conclusion that (3.1) has at least two solutions. ■

Theorem 3.2]
Suppose that the eigenvalue λ_{m0} is simple. Suppose that λ_1 is the nearest left eigenvalue of λ_{m0} and λ_2 is the nearest right eigenvalue of λ_{m0}. Then the equation

$$Lu + bu^+ = 1 + \varepsilon h \ in\ H$$

has at least two solutions for b in $(\lambda_1, \lambda_{m0})$ or in $(\lambda_{m0}, \lambda_2)$.

Proof]
The proof of this theorem is almost same as the proof of Theorem 3.1], Lemma 2.1] and Lemma 2.2]. Suppose that $\lambda_1 < \lambda < \lambda_2$. Then we can appply the Lemma 2.7] in each of the interval $(\lambda_1, \lambda_{m0})$ and $(\lambda_{m0}, \lambda_2)$. The degree is +1 in one of these intervals and -1 in the other interval. Since b must be one of these intervals, we can apply Theorem 2.1]. So we can reach the conclusion that the problem has at least two solutions for b in $(\lambda_1, \lambda_{m0})$ or in $(\lambda_{m0}, \lambda_2)$. ■

4. Nonlinear suspension bridge equation with a variable coefficient.

In this section we investigate what will happen if the coefficient $b = b(x)$.

$$u_{tt} + u_{xxxx} + bu^+ = 1 + \varepsilon f(x, t) \ in\ H \quad \text{...(4.1)}$$

We denote the set $\{\theta_{mn}\}$ is an orthonormal base in H.

Consider the problem

$$Lu + b(x)u^+ = 0 \ in\ H \quad \text{......(4.2)}$$

Let $H_1 = span\{\theta_{10}\}$ and $H_2 = H_1^\perp$. Let P denote the orthogonal projection on H_1.

Theorem 4.1]
Let $b(x)$ be even and $1.65 < b(x) < 3.35$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then the equation (4.2) has only the trivial solution.

Proof]
Let c=1.65. Then

$$\|(L+c)^{-1}\|_{H_1} = \frac{1}{1.35}$$
$$\|(L+c)^{-1}\|_{H_2} = \frac{1}{2.65}$$

Rewrite (4.2) as

$$(L+c)u + (b(x) - c)u^+ + cu^- = 0$$

Let $g(x) = b(x)u^+ + cu^-$ for all $u \in H$. Then we have

$$0 \le (b(x) - c)|u| \le g(x) \le 1.7|u|$$

and $\|g\| \le 1.7\|u\|$. Decompose g as $g = v + w,\ v = Pg,\ w = (I-P)g$. Then there exists $\delta (0 \le \delta \le 1)$ such that

$$\|v\|^2 = 1.7^2 \delta^2 \|u\|^2$$
$$\|w\|^2 \le 1.7^2 (1-\delta)^2 \|u\|^2 \quad \text{......(4.3)}$$

Since $v = \alpha \cos 2t \cos x$ for some $\alpha \in R$ and hence $\|v^+\|^2 = \frac{1}{2}\|v\|^2$. Thus we get

$$||v||^2 = \int gv = \int gv^+ - \int bv^-$$
$$\leq 1.7 \int |u| v^+ \leq \frac{1.7}{\sqrt{2}} ||u|| ||v|| \quad \text{...........(4.4)}$$

from which

$$||u||^2 \leq \frac{1.7^2}{2} ||u||^2 \quad \text{...............................(4.5)}$$

Then we get $0 \leq \delta^2 \leq \frac{1}{2}$. And

$$||(L+c)^{-1} g(x)||^2 < ||u||^2$$

Thus the equation

$$u + (L+c)^{-1} g(x) = 0 \quad \text{in } \boldsymbol{H}$$

has a unique solution, which is the trivial solution. ■

The proof of next lemmas are almost same as the proof in section 2 and section 3.

Lemma 4.1]
There exists $R_0 > 0$ such that for all $b(x)$ with $1.65 < b(x) < 3.35$ the solution u of (4.1) satisfy $||u|| < R_0$.

Lemma 4.2]
Under the same same assumptions in Lemma 4.1]

$$d_{LS}(u - L^{-1}(1 - b(x) u^+), B_R, 0) = 1$$

for large enough $R > R_0$.

Lemma 4.3]
Let $b(x)$ be even and $3 < b(x) < 3.35$. Then there exists $\gamma > 0$ such that

$$d_{LS}(u - L^{-1}(1 - b(x) u^+), B_\gamma(u), 0) = -1$$

where u is the positive solution of the equation (4.1).

Theorem 4.2]
Let $3 < b(x) < 3.35$ be even, then the equation (4.1) has at least 2 solutions.

5. Conclusion

In this paper, we investigate the number of solutions in the suspension bridge equation which is represented by a non-linearly supported vibrating beam. The case of an externally imposed period function is given and the coefficients are constant, we investigate how many solutions can be found in some range. To find the number of the solutions we used Leray-Schauder degree. And we investigate how many solution can be found when the equation contains variable coefficient.

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