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MAINTENANCE SETUP AND SETUP PERFORMANCE IMPROVEMENT IN AN UNRELIABLE PRODUCTION SYSTEM*

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ABSTRACT

An EOQ-like inventory model for a manufacturing process is studied. The system is assumed to deteriorate during the production process. The results are either the production of a number of defective items, or the breakdown of the production machine. The optimal production lot size is derived. The model is extended to the case in which the probabilities of making defective items and machine breakdowns are a function of both the quantity (amount) and quality (performance) of the consumed setup cost (including the preventive maintenance cost). We further assume that the setup performance can be improved by investing in the performance improvement program. Hence, the same or a better setup outcome can be achieved with a lower setup cost. We then investigate the optimal setup cost and investment policy simultaneously, thereby achieving a better process quality and setup cost reduction concurrently.

1. INTRODUCTION

Recent studies of Just-In-Time (JIT) systems have emphasized the use of setup reduction programs to achieve small production lot sizes and frequent deliveries, so as to reduce inventory carrying costs and increase the flexibility of the production system. However, one of the major impediments to the successful operation of such tightly coupled organizations is the breakdown of bottleneck resources or production of defective items.

In conjunction with this, Porteus [8] and Rosenblatt and Lee [10] have studied inventory systems, subject to the production of defective items. They model the stochastic process of making defective items very simply: while producing a lot, a process may go "out of control." Once the production process is in that state, it is assumed that it will produce defective items and

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continue to do so until the entire lot is finished. At the beginning of the next lot's production, the process is restored to the same initial in-control state. Both models analyze the policies of improving the reliability of the production system. However, the approaches differ significantly. Porteus [8] emphasizes setup reduction and process quality improvement programs so that his model simultaneously reduces setup cost and defective items by implementing the setup reduction program and investing in the process quality improvement program. On the other hand, Rosenblatt and Lee [10] focus on a more traditional preventive maintenance issue. They assume an explicit functional relationship between the probability of making defective items and setup cost (including the preventive maintenance setup), and then assume that the probability of making defective items can be reduced by increasing the setup cost.

In this article, a model has been designed to investigate the integrated maintenance setup policy and setup reduction program. Such an integration achieves both a reliable production process and smaller production lot sizes. The model differs from those considered by Porteus [8] and Rosenblatt and Lee [10] in two respects. First, Porteus [8] and Rosenblatt and Lee [10] considered a production system subject to the production of defective items only. On the other hand, we study a production system subject to both the production of defective items and machine breakdowns. Second, we use a different approach that reunifies the above mentioned two conflicting works, which both focus on a common goal of better quality. First, in our model, as in Rosenblatt and Lee [10], the probability of making defective items is reduced by increasing the preventive maintenance setup. Second, setup cost is reduced by investing in improvement to setup performance. Our model, therefore, interprets setup reduction differently than that of Porteus [6, 7, 8], and their related models [11, 1]. We regard a setup reduction program as an approach designed to improve setup performance as opposed to simply reducing setup cost. To see this, Sugimori, Kusunoki, Cho, and Uchikawa [12] have reported Toyota's success in reducing its setup time for a certain operation from one hour to ten minutes. However, needless to say, this does not mean that the outcome from the previous setup level also has been reduced by the same amount. Instead, a more precise statement might be that the setup performance has been improved by six times so that the same or a better outcome can now be achieved in 1/6 the previous setup time. This is precisely what we mean by reducing setup cost by improving the setup performance.

This paper is organized as follows. In § 2, assumptions and the stochastic nature of the production system are described. In § 3, an optimal lot size that accounts for defective items and machine breakdowns is specified. In § 4, the integrated preventive maintenance setup policy and setup performance improvement program are studied. Conclusions are provided in the last section.

2. PROBLEM DEFINITION

Let us assume a traditional EOQ world in which the basic model is modified as follows. First, while producing a single unit of product, equipment breakdown occurs with probability α $(1-\alpha=\beta)$. That is, the production system is assumed to follow a two-state Marcov chain during production, with a *transition* occurring with each unit produced. Once the breakdown takes place, the interrupted lot is aborted and the new one is started only when all available inventory is depleted. Equipment maintenance (including repair of the broken machine) is carried out after a failure or a predetermined inventory cycle time, whichever occurs first. The time for repair and setup is assumed to be negligible. Each maintenance action restores the system to the same initial working conditions. A similar assumption of machine breakdowns also has been used by Groenevelt, Pintelon, and Seidmann [2]. Their assumptions are that the production process can be interrupted by machine breakdowns, and that the time until the breakdowns is exponentially distributed.

Second, while in production, with probability q $(1-q=\rho)$, the production system begins to produce defective units. The stochastic process is similar to that for equipment breakdowns that is, it follows a two-state Marcov chain, with a transition occurring with each unit produced. Once the system is out of control, it remains that way until the remainder of the target lot has been completed or equipment breakdown takes place. Each defective item incurs an extra cost for reworking and related operations. Again, each maintenance action restores the system to the same initial working conditions. Note that the stochastic process of machine breakdown or production of defective items used in this article is different from the exponentially distributed elapse time models [2, 4, 5, 10]. Our assumption is akin to that of Porteus [8, 9], and Hong and Hayya [3] in which they assumed that

(1) The production rate is instantaneous.
(2) While producing a single unit of product, equipment can go "out-of-control" with probability q. That is, the production system is assumed to follow a two-state Marcov chain during production, with a transaction occurring with each unit produced. Once the equipment is out of control, it remains that way until the remainder of the target lot has

been completed.

Yano and Lee [13] comment that Porteus's representation of the time to failure is a discrete approximation of the exponential time to failure assumption. That is, the geometric distribution of the time to failure. Porteus [9] provides an example of such an operation.

> "...This Simple Modeling approach is supported by Moden (1983).... A high-speed automatic punch press, for example, where lots of 50 or 100 units are kept in a chute, only the first and the last unit in the chute are inspected. If both units are good, all units in the chute are considered good....."

The case of a finite production rate m is covered by adjusting $d = d'(m/(d'+m))$, where d' and m are demand and production rates respectively.

Figure 1 illustrates the resulting sample path for this model. For example, in the first inventory cycle, the target lot is accomplished, and the whole produced lot consists of good items. In the second cycle, the target lot is aborted due to machine failure. The produced lot consists of good items. In the third, target lot is achieved. However, a portion of the produced lot is defective. In the fourth cycle, a target lot is aborted due to machine failure,

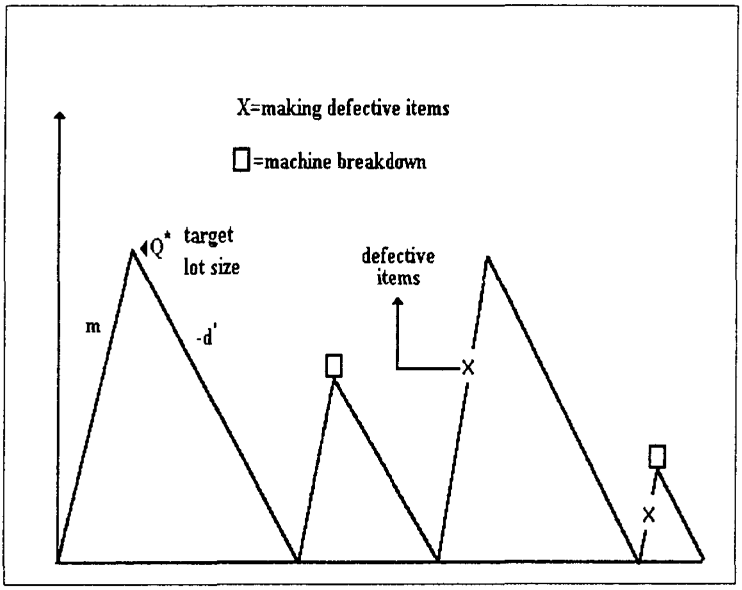

Figure 1. Inventory Sample Path

and a part of the produced lot consists of the defective items. We now introduce the notations used in this model.

Notation Summary for the Model

P :	cost of production
Q :	target lot size
$Z(Q)$:	expected lot size for a target lot size Q
δ_Z :	expected number of defective items in $Z(Q)$
r_1:	opportunity cost rate (Pr_1 holding cost)
r_s :	rework cost as a percentage of production cost
d':	demand rate, $d=d'$ when $m\to\infty$
S :	setup cost

We first develop the expected lot size $Z(Q)$ for a target lot size of Q , and the expected number of defective items in $Z(Q)$.

Lemma 1.
(1.1) The expected lot size is

$$Z(Q)=\frac{\beta(1-\beta^{Q})}{\alpha}, \tag{1}$$

We see that $Z(Q)<\beta/\alpha \quad \forall Q\in[0,\infty)$, *and* $\lim_{Q\to\infty} Z(Q)=\beta/\alpha$. *We also see that* $Z(Q)$ *is strictly concave and strictly increasing in* Q . *The expected shortfall of an aborted target lot due to machine breakdown* $Q-Z(Q)$ *is a strictly increasing, strictly convex function of* Q .

(1.2) The expected number of defective items is

$$\delta_Z=Z(Q)-\frac{\beta\rho(1-(\beta\rho)^{Q})}{1-\beta\rho} \tag{2}$$

which is strictly increasing in Q .

$$Y(Q)=\frac{\beta\rho(1-(\beta\rho)^{Q})}{1-\beta\rho}$$

is the expected number of good items; it is strictly concave and strictly increasing in Q .

Proof. The proof of Lemma 1 is given in Appendix 1.

Lemma 1.2 gives the expected number of defective items. For example, if q =0.01, α=0.001, the expected number of defective items in target lots sized 100 and 50 are, respectively, 29.05(29%) and 7.55(15%). If q decreases to 0.001, the expected number of defective items drops to 9.46(9.5%) and 2.47(4.9%). These figures illustrate how reducing the probability of making defective items or target lot size can significantly reduce the number of defective items. Porteus [8] has taken advantage of these two properties, and suggested two approaches to reduce the number of defectives. First, directly reduce q . Second, reduce the setup level to reduce the lot size; this leads to a smaller number of defectives. Consider now Lemma 1.1 and 1.2 simultaneously, and let q=0.01, α=0.01, and Q=100; then the expected numbers of lot size and defective items are 62.76 (62.76% of 100) and 20.11 (32% of 62.76), respectively. If α is decreased to 0.001, and Q is decreased to 65, then the expected numbers of lot size and defectives are 62.96 (97% of 65) and 16.84 (27% of 62.96). These figures tell us two things. First, a system with relatively high reliability (small α) incurs less shortfall ($Q-Z(Q)$) from an aborted target lot due to machine failure. Second, the percentage of defectives is smaller in a reliable system.

Figure 2 illustrates the expected number of defectives as a function of target lot size. It is interesting to know that contrary to the previously known

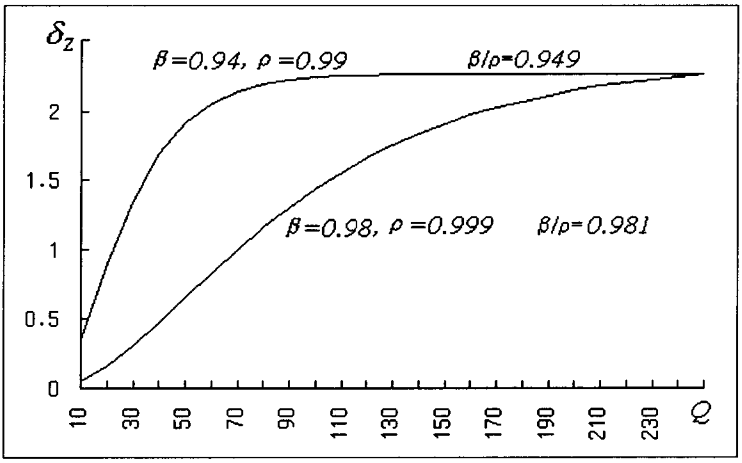

Figure 2. δ as a Function of Q

results (see, for example, Porteus [8]), the expected number of defective items is not always convex in Q. In particular, if β is relatively small (low reliability) and ρ is relatively large (high quality), then it is possible for δ_Z to be concave in Q. For an unreliable machine with a large target lot size, the actual lot size and number of defective items produced will almost always be determined by equipment breakdown.

3. OPTIMAL LOT SIZE

Here, we determine the optimal target lot size Q^* which minimizes the long-run average cost per unit time while accounting for the quality and reliability effects of this model. Note that the inventory process has a renewal epoch at the beginning of each inventory cycle due to setup operations. Therefore, using the well-known renewal reward theorem, the long-run average cost can be found by taking the ratio of the expected cost per renewal cycle and the expected duration of a renewal cycle. The long-run average cost function (3) consists of setup, holding, and rework costs.

$$\overline{C}(Q) = \frac{Sd}{Z(Q)} + \frac{Pr_1 Z(Q)}{2} + Pr_s d\left(1 - \frac{Y(Q)}{Z(Q)}\right) \tag{3}$$

An optimal target lot size Q^* minimizes cost expression (3). Proposition 1 summarizes the structural properties of the model.

Proposition 1. When α and q are very small:

(1.a) The total cost is approximately

$$\cong \frac{Sd}{Z(Q)} + \frac{Z(Q)P(r_1 + r_s dq/\beta^3)}{2} \tag{4}$$

(1.b) Optimal expected production lot size minimizing (4) is

$$Z(Q)^* = \sqrt{\frac{2Sd}{P(r_1 + r_s dq/\beta^3)}}.$$

By Lemma (1.1) the optimal target lot size is

$$Q^* = \frac{\ln(1-\alpha Z^*/\beta)}{\ln\beta}, \text{ if } Z^* < \beta/\alpha, \text{ and}$$

$$Q^* \to \infty \text{ as } Z^* \to \beta/\alpha \tag{5}$$

(1.c) If (5) exists, then the optimal cost is

$$\overline{C}(Q^*) = \overline{C}(Z^*) = \sqrt{2SdP(r_1 + r_s dq/\beta^3)} \tag{6}$$

Proof. Proofs for Proposition 1 are given in Appendix 2.

Proposition 1.b tells us that the optimal production lot size Q^* can be obtained by computing Z^* first, and then substituting into (5). Proposition 1.b also shows us that the optimal solution approaches infinity as Z^* approaches β/α. However, the model assumes α to be very small so that the ratio β/α is relatively large. Therefore, there is a relatively wide feasible area for the optimal solution.

Here, a numerical example given in Porteus [8] is provided to better understand the model : S =100, d =1000, r_1=0.15, r_s =0.5, q=0.0004, and P= 50. In addition five levels of β=0.995, 0.996, 0.997, 0.998, and 0.999 are considered. The accuracy of the approximation (obtained from (5)) is determined by testing against the classical EOQ model and the exact optimal solution, where the exact optimal solution is numerically searched from cost function (3). Total costs from the three approaches are all obtained from cost function (3).

Table 1. Comparison of the Optimal Solutions

β	0.995	0.996	0.997	0.998	0.999
Classical EOQ Q^*	163.3	163.3	163.3	163.3	163.3
Total Cost by (3)	2013	2003	1999	2002	2012
Optimal Q^* by (5)	152.7	139.3	128.8	120.2	114
Total Cost by (3)	1998	1964	1935	1910	1888
Exact Optimal	130.0	124.7	120.0	115.7	114
Total Cost by (3)	1982	1956	1931	1909	1888

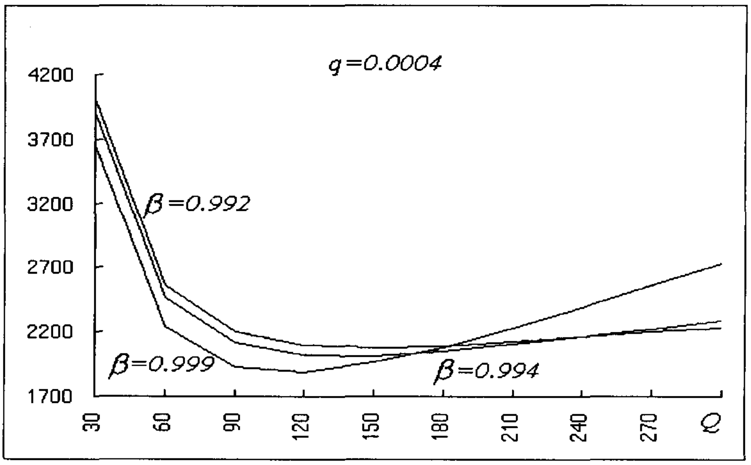

Figure 3. Long-Run Average Cost as a Function of Lot Size

Table 1 shows that the total costs and target lot sizes obtained from (5) and the exact optimal solution are almost identical when β is relatively large. Through the numerical experiment we discovered an interesting property.

Figure 3 pictures the total cost as a function of the targeted lot size Q for β =0.992, 0.994, 0.999. For a smaller value of β , the optimum value of Q shifts to the right. This property also has been verified by Groenevelt, Pintelon, and Seidmann [2] in a similar model with a different set of assumptions about the machine breakdown process. In addition, as in Groenevelt et al., the cost function flattens out for a smaller value of β (unreliable machine). With an unreliable machine, the actual lot size will almost always be determined by equipment breakdown. Therefore, when all other factors are kept unchanged, the cost function is almost indifferent to the target lot size after it passes a certain number.

4. MAINTENANCE SETUP AND SETUP PERFORMANCE IMPROVEMENT

We assumed in the previous sections that the probabilities of making defective items or of machine breakdowns are independent of setup costs. However, different levels of setups (including preventive maintenance), as discussed in Rosenblatt and Lee [10], may affect the production process. In this section, we will investigate the simultaneously reduction of the probability of making defective items by increasing the setup cost and investing in improving the setup performance, thereby achieving better process quality and setup cost reduction concurrently. For the sake of simplicity, we redefine $q=q/\beta^3$ so that

q is now the ratio of the probability of making defective items to one subtract probability of machine breakdowns. Hereafter, we shall assume q to be functions of k (setup performance) and S (setup cost). Let f_x denote the differentiation of f by x. We assume

(i) q and α are very small so that $0 \leq q(k, S) \leq 1$, and

(ii) $q_s < 0$, $q_{ss} \geq 0$, $q_k < 0$, and $q_{kk} \geq 0$.

We let $\phi(k, k_0)$ be the investment cost of improving the setup performance from the original level k_0 to level k. We assume $\phi_k > 0$, and $k \in [k_0, \infty)$.

In general, we seek to minimize

$$\min_{k,S} \ \omega(Q^* k, S) = \overline{C}(Q^*, k, S) + \phi(k, k_0) \tag{7}$$

Then,

$$\omega_s = 0 \ \Rightarrow \ \frac{(r_1 + q r_s d)}{S r_s d} + q_s = 0 \tag{8}$$

$$\omega_k = 0 \Rightarrow \frac{\phi_k \sqrt{2SdP(r_1 + q r_s d)}}{S d r_s d P} + q_k = 0 \tag{9}$$

A simultaneous solution may be derived for some specification of $q(k, S)$ and $\phi(k, k_0)$. For example consider the form

(i) $q(k, S) = \tilde{q}(kS/k_0 S_0)^{-b}$, where $S \in [S_0, \infty)$, and $b > 1$. Here, $S_0 > 0$ is the minimum level of setup cost that must be sustained in the production process, while $\tilde{q} = q_0 / \beta_0^3$, where q_0 and β_0 are the original probability levels.

(ii) $\phi(k, k_0) = b_k r_1 \ln(k/k_0)$, where $b_k r_1 \ln(k/k_0)$ is the investment cost of changing k value from the original k_0 to the improved level k, and b_k is the costs of making an approximately 63% reduction in $1/k$. One reasonable explanation for this cost function is to regard $1/k$ as an index of "ineffective performance" level; the investment is made to rectify it. This investment cost function is well documented in

Porteus[6], so the explanatory details are omitted here Proposition 2 summarizes the optimal decision rules.

Proposition 2.

(2.a) *$\omega(Q^*, k, S)$ has a unique local minimum k^* on $[k_0, \infty)$ and S on $[S_0, \infty)$. The optimal setup cost and setup performance level satisfying (8) and (9) are*

$$S(k) = \left(\frac{\tilde{q} r_s d(b-1)}{r_1}\right)^{\frac{1}{b}} \left(\frac{S_0 k_0}{k}\right) \tag{10}$$

and
$$k(S) = \left[\frac{d\tilde{q} r_s \left(\sqrt{1 + 2\left(\frac{P r_1 S d b^2}{(b_k r_1)^2}\right)} - 1\right)}{2 r_1}\right]^{\frac{1}{b}} \left(\frac{k_0 S_0}{S}\right) \tag{11}$$

For the purposes of the next proposition, let

$$S1 = \sqrt{\frac{P r_s S_0 \tilde{q}}{2(r_1 / d\tilde{q} r_s + 1)} \frac{db}{r_1}}, \quad S2 = \sqrt{\frac{S_0 d b P}{2 r_1 (b-1)}},$$

$$S3 = \left[\frac{\tilde{q} r_s d(b-1)}{r_1}\right], \text{ and } S4 = \sqrt{\left(\frac{\tilde{q} r_s d(b-1)}{r_1}\right)^{\frac{1}{b}} \frac{P d b S_0}{2(b-1) r_1}}$$

We now introduce four different cases that emerged through this process. Let

$C1: = \{b_k \geq S2, b_k \leq S4\}$, $C2: = \{S3 \geq 1, b_k > S4\}$,
$C3: = \{b_k < S2, b_k \leq S1\}$, and $C4: = \{S3 < 1, b_k > S1\}$.

(2.b) $C1$, $C2$, $C3$, and $C4$ are mutually exclusive and collectively exhaustive.

(2.c) The optimal solution satisfies one of the following four cases.
Case $C1$: $C1$ holds if and only if

$$S^*(k^*) = \frac{2(b_k r_1)^2 (b-1)}{d b P r_1},$$

$$k^*(S^*) = \left(\frac{\tilde{q} r_s d(b-1)}{r_1}\right)^{\frac{1}{b}} \left(\frac{k_0 S_0 d b P r_1}{2(b-1)(b_k r_1)^2}\right).$$

S^* and k^* are obtained by simultaneously solving for (10), (11).

Case C2: C2 holds if and only if

$$S^*(k_0) = S_0\left(\frac{\tilde{q} r_s d(b-1)}{r_1}\right)^{\frac{1}{b}}, \; k^* = k_0,$$

S^* is obtained by substituting k_0 into (10).

Case C3: C3 holds if and only if

$$S^* = S_0, \; k^*(S_0) = k_0\left(\frac{d\tilde{q} r_s\left(\sqrt{1+2\left(\frac{P r_1 S_0 d b^2}{(b_k r_1)^2}\right)}-1\right)}{2r_1}\right)^{\frac{1}{b}}.$$

k^* is obtained by substituting S_0 into (11).

Case C4: C4 holds if and only if $S^* = S_0$, $k^* = k_0$.

Proof. The proof for Proposition 2 is given in Appendix 3.

Through (10) and (11), we see that both S and k are increasing in $\tilde{q}$, and r_s, and decreasing in r_1. We also see that k is decreasing in b_k. The four cases specified in Proposition 2 correspond to the following four policies, respectively: Case C1, maintaining the optimum setup level and at the same time investing in improving the setup performance; Case C2, maintaining the optimum setup level only; Case C3, maintaining the minimum setup level and at the same time investing in improving the setup performance; and Case C4, maintaining the minimum setup level without investing in improving the setup performance. One can interpret the option of setup reduction in connection with Proposition 2 as follows: Before the improvement program (setup reduction program), the setup cost and setup performance are given in Case C2. With the improvement program, the setup level (performance) is reduced (improved) to Case C1. For example consider a numerical example: $P=1$, $S_0=100$, $d=1000$, $b=2$, $r_1=0.15$, $r_s=0.5$, $k_0=100$, and $\tilde{q}=(0.005/0.999^3)$. In addition, three b_k levels are considered: b_k =10,000, 3000, and 30,000. Upon substitution, the four indices are, respectively, $S1$=32427.5, $S2$=5773.5, $S3$ =16.7, and $S4$=11674. We see that three b_k levels imply three different cases: b_k =10,000 implies case $C1$, b_k =3,000 implies case $C3$, and b_k =30,000 implies case $C2$. Note that ρ can be computed by $\rho = 1 - q^* \beta^3$.

The results summarized in Table 2 show three optimal solutions relating to three levels of b_k. For example, when b_k=30,000, the optimal policy is to increase the setup level to 409 and maintain the setup performance to the minimum level. Doing so makes perfect sense since the cost of improving the setup performance is very expensive in this case. Table 2 also shows that the lot size for case $C3$ is always less than cases $C1$ and $C2$, and more than $C4$.

Table 2. Results of the Numerical Example

b_k=10000	$C1$	$C2$	$C3$	$C4$	EOQ
S^*	300	408.9	100	100	100
k^*	136.3	100	276.5	100	100
q^*	0.0003	0.0003	0.0007	0.005	0.005
Q^*	223.3	266.0	95.99	39.6	163.3
Z^*	200.0	233.5	91.5	38.8	150.6
$100(\delta_Z/Z^*)$	3.16	3.72	3.06	9.47	31.1
ω^*	3504	3557	3726	5090	9013
b_k=3000					
S^*	-	408.9	100	100	100
k^*	-	100	615.6	100	100
q^*	-	0.0003	0.00013	0.005	0.005
Q^*	-	266.0	146.3	39.6	163.3
Z^*	-	233.5	136	38.8	150.6
$100(\delta_Z/Z^*)$	-	3.72	0.94	9.47	31.1
ω^*	-	3557	2300	5090	9013
b_k=30000					
S^*	-	408.9	100	100	100
k^*	-	100	107.6	100	100
q^*	-	0.0003	0.0043	0.005	0.005
Q^*	-	266.0	42.48	39.6	163.3
Z^*	-	233.5	41.56	38.8	150.6
$100(\delta_Z/Z^*)$	-	3.72	8.79	9.47	31.1
ω^*	-	3557	5089	5090	9013

The reason for its being less than $C1$ and $C2$ is apparent. As for more than $C4$, the reason is that a system with lower process quality (higher probability of making defective items) requires a smaller production lot size to reduce the average fraction defective δ_Z/Z^* and the total cost. This property also has been discussed in Porteus [8].

5. CONCLUSION

An EOQ-like inventory model for a manufacturing process was studied. The system is assumed to deteriorate during the production process. The results are either the production of a number of defective items, or the breakdown of the production machine. The optimal production lot size is derived. The model was extended to the case in which the probabilities of making defective items and machine breakdowns are a function of both the quantity (amount) and quality (performance) of the consumed setup cost (including preventive maintenance cost). We further assume that the maintenance setup performance can be improved by investing in the setup improvement program. Hence, the same or a better setup outcome can be achieved with a lower setup cost.

In conjunction with this, an integrated maintenance setup and setup performance improvement policy was studied. The approach improves process quality by increasing the setup level while simultaneously reducing it by improving setup performance, thereby achieving both the improvement of process quality and the reduction of setup cost. The economic production lot sizes, investment policies, and properties for the model are provided. Finally, a numerical example is provided to explain the optimal policies.

Appendix 1: Proof of Lemma 1

Lemma 1.1. The expected production lot size is

$$\alpha\sum_{k=0}^{Q-1} k\beta^k + \alpha\sum_{k=Q}^{\infty} Q\beta^k = \alpha\left[\sum_{k=0}^{Q-1} k\beta^k + \sum_{k=0}^{\infty} Q\beta^k - \sum_{k=0}^{Q-1} Q\beta^k\right]$$

$$= Q-(Q-Z(Q)) = Z(Q).$$

The results follow since $Z'(Q) = -(\beta^Q \ln\beta)\beta/\alpha > 0$. Further differentiation reveals that the second derivative is strictly negative. Since $-\beta^Q\ln\beta \leq -\beta\ln\beta < 1-\beta$, we see that $Q-Z(Q)$ is a strictly increasing function of Q. The second differentiation reveals that the second derivative is strictly positive, which proves the result.

Lemma 1.2. The expected number of defective items is

$$\delta_Z = \alpha\Big[\sum_{k=0}^{Q-1}\beta^k\Big(qk\sum_{i=0}^{k-1}\rho^i - q\sum_{i=1}^{k-1}i\rho^i\Big) + \sum_{k=Q}^{\infty}\beta^k\Big(qQ\sum_{i=0}^{Q-1}\rho^i - q\sum_{i=1}^{Q-1}i\rho^i\Big)\Big]$$

$$= \alpha\Big[\sum_{k=0}^{Q-1}\beta^k\Big(k - \frac{\rho(1-\rho^k)}{q}\Big) + \Big(Q - \frac{\rho(1-\rho^Q)}{q}\Big)\Big(\sum_{k=0}^{\infty}\beta^k - \sum_{k=0}^{Q-1}\beta^k\Big)\Big]$$

$$= \alpha\Big[\Big(\sum_{k=0}^{Q-1}\beta^k k - \sum_{k=0}^{Q-1}\beta^k Q\Big) + \frac{\rho}{q}\Big(\sum_{k=0}^{Q-1}(\beta\rho)^k - \sum_{k=0}^{Q-1}\beta^k\rho^Q\Big) + \Big(Q - \frac{\rho(1-\rho^Q)}{q}\Big)\sum_{k=0}^{\infty}\beta^k\Big]$$

$$= \Big(\frac{\beta(1-\beta^Q)}{\alpha} - Q\Big) + \Big(Q - \frac{\beta\rho(1-(\beta\rho)^Q)}{1-\beta\rho}\Big).$$

Direct differentiation reveals that

$$\partial\delta_Z/\partial Q = \frac{(\beta\rho)^Q\beta\rho\ln\beta\rho}{1-\beta\rho} - \frac{\beta\beta^Q\ln\beta}{1-\beta}$$

$$= \frac{\beta^{Q+1}}{(1-\beta\rho)(1-\beta)}(\rho^{Q+1}\ln\beta(1-\beta) + \rho^{Q+1}\ln\rho(1-\beta) - \ln\beta(1-\beta\rho)).$$

We see that $\partial\delta_Z/\partial Q > 0$, since

$0 > \frac{\rho^{Q+1}\ln\rho}{(1-\beta\rho)-(1-\beta)\rho^{Q+1}} > \frac{\rho\ln\rho}{1-\rho} > -1 > \frac{\ln\beta}{1-\beta}$, which proves the result.

It easily follows, upon differentiation, that the sign of $Y'(Q) > 0$ and $Y''(Q) < 0$. □

Appendix 2: Proof of Proposition 1

Proposition 1.a. When α is very small, $1-\beta^x \cong -(\ln\beta)x - [(\ln\beta)x]^2/2$. We also see that when $\beta \geq 1/2$, $\ln\beta = (-\alpha/\beta) + (-\alpha/\beta)^2/2 + (-\alpha/\beta)^3/3 + \ldots$. Assuming α is very small, we use the first term in the approximation. This approach also was used by Porteus [8,9]. Thus, $Z(Q) \cong Q[1-\alpha Q/2\beta]$. Similarly, when q and α are very small, $Y(Q) \cong Q[1-(1-\beta\rho)Q/2\beta\rho]$. Therefore,

$$\delta_Z/Z \cong 1 - \frac{Y(Q)}{Z(Q)} \cong \frac{Q^2 q}{2\beta\rho Z(Q)} \qquad \text{(P1a.1)}$$

By linear approximation $Z(Q) = \beta(1-\beta^Q)/\alpha \cong -\beta Q\ln\beta/\alpha$. Now, given that α is very small $-\beta\ln\beta/\alpha = -\beta((-\alpha/\beta) + (-\alpha/\beta)^2/2 + \ldots)/\alpha \cong 1 - \alpha/2\beta \cong \beta$,

Table 4. Comparisons of γ and β

β	0.99	0.992	0.994	0.996	0.998	0.999	0.9999
γ	0.995	0.996	0.997	0.998	0.999	0.9995	0.99995
% of Difference	0.503	0.402	0.301	0.201	0.100	0.050	0.005

Table 5. Comparisons of δ_Z/Z and $Zq/2\beta^3\rho$ (β=0.999 ρ=0.999)

Q	10	20	30	40	50	60	70	80
δ_Z/Z	0.548	1.04	1.528	2.011	2.489	2.962	3.430	3.894
$Zq/2\beta^3\rho$	0.499	0.994	1.483	1.967	2.448	2.922	3.392	3.858
% of Difference	8.82	4.46	2.94	2.15	1.67	1.34	1.11	0.94

and so $Z(Q)\cong Q\beta$. Substituting the approximation into (P1a.1) gives $\delta_Z/Z\cong Zq/2\beta^3\rho$. Finally, as in Porteus [8], the result follows upon using $\rho\cong 1$. Table 4 gives $\gamma=-\beta\ln\beta/a$ as a function of β. Table 5 compares δ_Z/Z and $Zq/2\beta^3\rho$. It tells us that the approximation approaches the exact solution as the target lot size increases.

Propositions 1.b-1.c. These results were derived after differentiation and substitution.

Appendix 3: Proof of Proposition 2

Proposition 2.a (10) and (11) follow from (8) and (9), and particular forms of $q(k,S)$ and ϕ_k.

$$\text{Let}\quad \widetilde{\omega_s}=\omega_s S^b \quad =\frac{\left(r_1 S^b+\tilde{q}\left(\frac{k}{k_0 S_0}\right)^{-b} r_s d\right)}{r_s d}-b\tilde{q}\left(\frac{k}{k_0 S_0}\right)^{-b}$$

We see that $\widetilde{\omega_s}$ has the same sign as ω_s. 2.a follows since,

(i) $\lim_{S\to 0}\widetilde{\omega_s}(S)<0$ and $\lim_{S\to\infty}\widetilde{\omega_s}(S)>0$, and

(ii) $\widetilde{\omega_s}(S)$ is a strictly increasing function of S.

By the same argument, we let

$$\widetilde{\omega_k}(k)=\omega_k(k)\,k^{2b}=-\left(\frac{k_0 S_0}{S}\right)^{2b}$$

$$+2\left(\frac{\Omega}{r_s\tilde{q}}\right)^2\left(\frac{r_1}{Sd}\right)k^{2b}+2\left(\frac{(k_0S_0)^b\Omega^2}{r_s\tilde{q}S^{1+b}}\right)k^b=0,$$

the result follows similarly. □

Proposition 4.b Here, we show $C3$ to be disjointed from $C1$, $C2$, and $C4$. Using a similar argument, it is not difficult to show the remaining cases. It can be easily seen that the pairs $(C1, C3)$ and $(C4, C3)$ are disjointed. We now proceed to show that $C2$ and $C3$ are disjointed. $S4=S2\sqrt{S3}^{1/b}$; hence, $b_k < S2$ and $b_k > S4$ imply $S3 < 1$, which contradicts $C2$.

Proposition 4.c The four cases are developed based on whether investment is made to improve setup performance and whether the setup level should be increased. For example, consider Case $C3$, $S^* < S_0$ and $k^* \geq k_0$. For this to be optimal, $k^*(S_0) \geq k_0$, and $S^*(k^*) < S_0$. The first condition is equivalent to

$$k_0\left(\frac{d\tilde{q}r_s\left(\sqrt{1+2\left(\frac{Pr_1S_0db^2}{(b_kr_1)^2}\right)}-1\right)}{2r_1}\right)^{\frac{1}{b}} \geq k_0$$

which leads to $b_k \leq S1$. The second condition is equivalent to

$$\left(\frac{\tilde{q}r_sd(b-1)}{r_1}\right)^{\frac{1}{b}}\left(\frac{S_0k_0}{k}\right) < S_0$$

which, after substituting for $k^*(S_0)$, leads to $b_k < S2$. This completes the condition of $C3$. The other cases follow similarly. Their sufficiency follows from their mutual exclusivity as verified in 2.b.

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