論 文 12-4-28

DQM을 이용한 아치의 좌굴해석

Buckling Analysis of Arches Using DQM

한 지 원* · 강 기 준*
Ji-Won Han · Ki-Jun Kang
(1997년 8월 12일 접수, 1997년 12월 12일 채택)

ABSTRACT

The differential quadrature method (DQM) is applied to computation of the eigenvalues of the equations governing in plane and out-of-plane buckling. In-plane buckling and twist-buckling under uniformly distributed radial loads are investigated by this method. Critical loads are calculated for various end conditions and opening angles. Results are compared with existing exact solutions where available. The differential quadrature method gives good accuracy even when only a limited number of grid points is used. New results are given for two sets of boundary conditions not previously considered for this problem : clamped-clamped and clamped simply supported ends.

국 문 요 약

곡선보(curved beam)의 등분포 하중하에서 평면내(in-plane)와 평면외(out-of-plane)의 좌굴(buckling)을 해석하는데 differential quadrature method (DQM)을 이용하여 다양한 경계조건(boundary conditions)과 굽힘각(opening angles)에 따른 임계하중(critical loads)을 계산하였다. DQM의 해석결과는 해석적 해답(exact solution) 또는 다른 수지해석 결과와 비교하였으며, DQM은 적은 요소(grid points)를 사용하여 정확한 해석결과를 보여주었다. 두 경계조건(고정-고정, 고정-단순지지)하에서 새로운 결과를 또한 제시하였다.

1. Introduction

Owing to their importance in many fields of technology and engineering, the stability behavior of elastic arches been the subject of a large number of investigations. Solutions of

* 호서대학교 기계공학부

the relevant differential equations have traditionally been obtained by the standard finite difference or finite element methods. These techniques require a great deal of computer time as the number of discrete nodes becomes relatively large under conditions of complex geometry and loading. In a large number of cases, the moderately accurate solution which can be calculated rapidly is desired at only a few points in the physical domain. However, in order to get results with even only limited accuracy at or near a point of interest for a reasonably complicated problem, solutions often have dependence of the accuracy and stability of the mentioned methods on the nature and refinement of the discretization of the domain.

Ojalvo et al.[1)] studied the elastic stability of ring segments with a thrust or a pull directed along the chord. Vlasov[2)] derived closed-form solutions such as for an arch, in which cross-sections are allowed to warp non-uniformly along the beam axis, subject to in-plane bending and uniformly distributed radial loads. Cheney[3)] studied the buckling of thin-walled open-section rings including both the effect of axial stress and the effect of warping. Yoo and Pfeiffer[4)] derived the flexural-torsional buckling equations which were based on a different derivation of the total potential. Papangelis and Trahair[5)] conducted a theoretical study of the flcxural-torsional buckling of doubly symmetric arches to confirm the predictions of Timoshenko and Gere[6)] for arches in uniform compression and of vlasov[2)] for arches in uniform bending. Trahair and Papangelis[7)] also developed a flexural-torsional buckling theory for arches of monosymmetric cross-section using the second variation of the total potential. Yang and Kuo[8)] studied the static stability of curved thin-walled beams using the principle of virtual displacements in a Lagrangian formulation with emphasis place on the effect of curvature, and they presented closed-form solutions for arches in uniform bending and uniform compression. In addition, different approaches were also presented by Kuo and Yang[9)] to support their studies treating a curved beam as the composition of an infinite number of infinitesimal straight beams. Recently, Kang and Yoo[10)] presented a theoretical study on the buckling of thin-walled curved beams with the derivation of stability equations. Very recently Pi et al.[11)] investigated the effect of prebuckling deformations on flexural-torsional buckling of arches, and Kang and Bert[12)] studied flexural-torsional buckling with warping using the differential quadrature method (DQM).

In the present work, the DQM which is a rather efficient alternate procedure for the solution of partial differential equations, introduced by Bellman and Casti[13)], is use to analyze in-plane buckling and twist-buckling under uniform pressure. The critical loads are calculated for the member. The circular arches considered are of uniform cross section, and have both ends either simply supported or clamped, or have simply supported-clamped ends, Numerical results are compared with existing exact solutions where available.

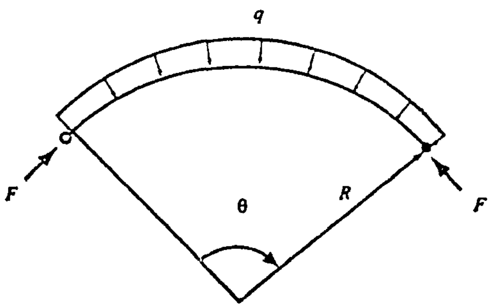

Fig. 1 Uniformly distributed radial loading

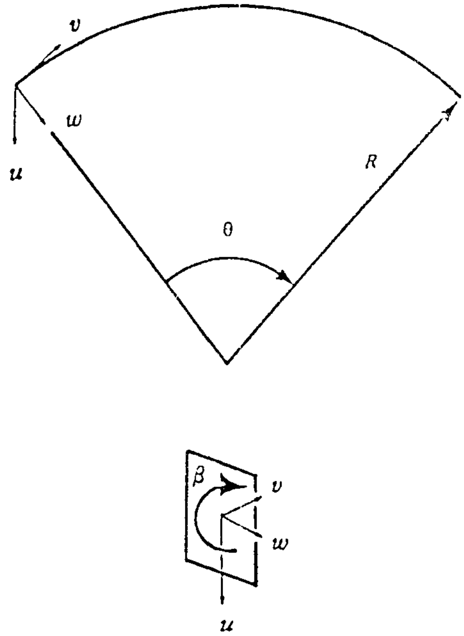

Fig. 2 Coordinate system of arch

2. Governing Differential Equations

The uniform circular arch considered is shown in Figs. 1 and 2 under a uniform in ward radial pressure q per unit of cir-cumferential length. A point on the centroidal axis is defined by the angle θ, measured from the left support, and the radius of the centroidal axis is R. The tangential and radial displacements of the arch axis are v and w, respectively, u and β are the displacement at right angles to the plane of the arch and the angular rotation of a cross section, respec-tively. These displacements are considered to be positive in the directions indicated.

2.1 In-plane buckling of uniformly compressed thin circular arches

A mathematical study of the in plane inex-tensional condition of small cross section is carried out starting with the basie equations as given by Love[14]. Following Love, the analysis is simplified by restrieting attention to problems where there is no extension of the center line. This condition requires that w and v be related by

$$w=-\frac{\partial v}{\partial \theta} \quad \cdots\cdots (1)$$

If the external forces are assumed to rotate with the centroidal axis of the arch during the process of buckling, and shear deformation is neglected, the differential equation can be written as

$$-\frac{EI}{R^4}(w^{V}+w''')+\frac{EI}{R^4}(v^{VI}+v'') = \frac{q}{R}(w'''+w') \quad \cdots\cdots (2)$$

where E is the Young's modulus of clasticity for the material of the arch, and I is the area moment of inertia of the cross section.

Using equation (1) and the dimensionless distance coordinate $X=\theta/\theta_0$, in which θ_0 is the opening angle of the member, one can rewrite equations (2) as

$$\frac{v^{VI}}{\theta_0^6}+2\frac{v^{VI}}{\theta_0^4}+\frac{v''}{\theta_0^2} = -\left(\frac{qR^3}{EI}\right)\left(\frac{v^{VI}}{\theta_0^4}+\frac{v''}{\theta_0^2}\right) \quad \cdots\cdots (3)$$

where each prime denotes one differentiation with respect to the dimensionless distance coordinate.

The boundary conditions for both ends clamped, both ends simply supported and for mixed clamped-simply supported ends are, respectively

$v=v'=v''=0$ at $X=0$ and 1 ……… (4)

$v=v'=v'''=0$ at $X=0$ and 1 ……… (5)

$v=v'=v''=0$ at $X=0$,

$v=v'=v'''=0$ at $X=1$ ……… (6)

2.2 Twist-buckling of uniformly compressed thin circular arches

Consider a circular arch shown in Fig. 1. The compressive force F in the arch is qR. This compressive force may cause buckling of the arch either in its plane or out of its plane. The corresponding buckling equations can be diduced from the coupled twist-bending vibration equations suggested by Timoshenko in investigating the torsional buckling of open section columns. His procedure is merely to replace the external load term by a fictitious load whose intensity is the load causing buckling times the appropriate 'curvature' term. On this basis, the buckling equations can be deduced from the equations of vibrations by formally replacing the inertia term. The differential equation can be written as (Wah[15])

$$\frac{EI}{R^4}\left(\frac{u^{\mathrm{VI}}}{\theta_0^4}-R\frac{\beta''}{\theta_0^2}\right)-\frac{GJ}{R^4\theta_0^2}(u''+R\beta'')=-\frac{q}{R}\frac{u''}{\theta_0^2} \quad \cdots\cdots (7)$$

$$\frac{EI}{R^2}\left(-R\beta+\frac{u''}{\theta_0^2}\right)+\frac{GJ}{R^2\theta_0^2}(u''+R\beta'')=0 \quad \cdots\cdots (8)$$

where G is the shear modulus, and J is the torsion constant of the cross section.

The boundary conditions for both ends clamped, both ends simply supported and for mixed clamped-simply supported ends are, respectively

$\beta=u=u'=0$ at $X=0$ and 1 ………………… (9)

$\beta=u=u''=0$ at $X=0$ and 1 ……………… (10)

$\beta=u=u'=0$ at $X=0$,

$\beta=u=u''=0$ at $X=1$ ……………………… (11)

3. Differential Quadrature Method

The Differential Quadrature Method was introduced by Bellman and Casti[13]. By formulating the quadrature rule for a derivative as an analogous extension of quadrature for integrals in their introductory paper, they proposed the differential quadrature method as a new technique for the numerical solution of initial value problems of ordinary and partial differential equations. It was applied for the first time to static analysis of structural components by Jang et al.[16]. The versatility of the DQM to engineering analysis in general and to structural analysis in particular is becoming increasingly evident by the related publications of recent years. Kukreti et al.[17] calculated the fundamental frequencies of tapered plates, and Farsa et al.[18] applied the method to analysis and detailed parametric evaluation of the fundamental frequencles of general anisotropic and laminated plates. In another development, the quadrature method was introduced in lubrication mechanics by Malik and Bert[19]. Kang and Bert[12] applied the method to the flexural-torsional buckling analysis of circular arches. From a mathematical point of view, the application of the differential quadrature method to a partial differential equation can be expressed as follows:

$$L\{f(x)\}_i=\sum_{j=1}^{N}W_{ij}f(x_j) \quad \text{for } i,\ j=1,2,\cdots,N \quad \cdots\cdots (12)$$

where L denotes a differential operator, x_j are the discrete points considered in the domain, $f(x_j)$ are the function values at these points, W_{ij} are the weighting coefficients attached to these function values, and N denotes the number of discrete points in the domain. This equation, thus, expresses as the derivatives of a function at a diserete point in terms of the function values at all discrete points in the variable domain.

The general form of the function $f(x)$ is taken as

$f_k(x)=x^{k\ 1}$ for $k=1,2,3,\cdots,N$ ………… (13)

If the differential operator L represents an n^{th} derivative, then

$$\sum_{j=1}^{N} W_{ij}\ x_j^{k\ 1} = (k-1)(k-2)\cdots(k-n)x_j^{k\ n-1}$$

for $i,\ k=1,2,\cdots,N$ .... (14)

This expression represents N sets of N linear algebraic equations, giving a unique solution for the weighting coefficients, Wij, since the coefficient matrix is a Vandermonde matrix which always has an inverse, as described by Hamming[20]. Thus, the method can be used to express the derivatives of a function at a discrete point in terms of the function values at all discrete points in the variable domain.

4. Application

The DQM is applied to the determination of the in-plane buckling and the out-of-plane buckling of circular arches. The differential quadrature approximations of the governing equations and boundary conditions are shown.

4.1 In-plane buckling of uniformly compressed thin circular arches

Applying the differential quadrature method to equations (3), gives

$$\frac{1}{\theta_0^6}\sum_{j=1}^{N} F_{ij}\ v_j + \frac{2}{\theta_0^4}\sum_{j=1}^{N} D_{ij}\ v_j + \frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\ v_j = -\left(\frac{qR^3}{EI}\right)\left(\frac{1}{\theta_0^4}\sum_{j=1}^{N} D_{ij}\ v_j + \frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\ v_j\right) \quad \text{.............. (15)}$$

where B_{ij}, D_{ij} and F_{ij} are the weighting coefficients for the second-, fourth- and sixth-order derivatives, respectively, along the dimensionless axis.

The boundary conditions for clamped ends, given by equations (4), can be expressed in differential quadrature form as follows :

$$v_1 = 0 \qquad at\ X=0$$
$$v_N = 0 \qquad at\ X=1$$

$$\sum_{j=1}^{N} A_{2j}\ v_j = 0 \qquad at\ X=0+\delta$$
$$\sum_{j=1}^{N} A_{(N-1)_j}\ v_j = 0 \quad at\ X=1-\delta$$
$$\sum_{j=1}^{N} B_{3j}\ v_j = 0 \qquad at\ X=0+2\delta$$
$$\sum_{j=1}^{N} B_{(N-2)_j}\ v_j = 0 \quad at\ X=1-2\delta \quad \text{.............. (16)}$$

Similarly, the boundary conditions for simply supported ends given by equations (5) can be expressed in differential quadrature form as follows :

$$v_1 = 0 \qquad at\ X=0$$
$$v_N = 0 \qquad at\ X=1$$

$$\sum_{j=1}^{N} A_{2j}\ v_j = 0 \qquad at\ X=0+\delta$$
$$\sum_{j=1}^{N} A_{(N-1)_j}\ v_j = 0 \quad at\ X=1-\delta$$
$$\sum_{j=1}^{N} C_{3j}\ v_j = 0 \qquad at\ X=0+2\delta$$
$$\sum_{j=1}^{N} C_{(N-2)_j}\ v_j = 0 \quad at\ X=1-2\delta \quad \text{.............. (17)}$$

where A_{ij} and C_{ij} are the weighting coefficients for the first- and third-order derivatives. here, δ denotes a very small distance measured along the dimensionless axis from the boundary ends. In their work on the application of DQM to the static analysis of beams and plates, Jang et al.[16] proposed the so-called δ-technique wherein adjacent to the boundary points of the differential quadrature grid points chosen at a small distance. This δ qpproach is used to apply more than one boundary conditions at a given station.

The boundary conditions for one clamped and one simply supported end, given by equations (6), can be expressed in differential quadrature form as

$$v_1 = 0 \qquad at\ X=0$$
$$v_N = 0 \qquad at\ X=1$$

$$\sum_{j=1}^{N} A_{2j}\, v_j = 0 \quad at \quad X = 0 + \delta$$
$$\sum_{j=1}^{N} A_{(N-1),}\, v_j = 0 \quad at \quad X = 1 - \delta$$
$$\sum_{j=1}^{N} B_{3j}\, v_j = 0 \quad at \quad X = 0 + 2\delta$$
$$\sum_{j=1}^{N} C_{(N-2),}\, v_j = 0 \quad at \quad X = 1 - 2\delta$$

.... (18)

Mixed boundaries can be casily accommodated by combining these equations ; simply change the weighting coefficients. While most analytical methods use the rather laborious technique of superposition to arrive at solutions for mixed boundary problems, this approach of breaking the problem into several easy is not required in the DQM. This set of equations together with the appropriate boundary conditions can be solved for the in-plane buckling.

4.2 Twist-buckling of uniformly compressed thin circular arches

Applying the differential quadrature method to equations (7) and (8), gives

$$\frac{EI}{R^4}\left(\frac{1}{\theta_0^4}\sum_{j=1}^{N} D_{ij}\, u_j - \frac{R}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\,\beta_j\right) - \frac{GJ}{R^4\theta_0^2}\left(\sum_{j=1}^{N} B_{ij}\, u_j + R\sum_{j=1}^{N} B_{ij}\,\beta_j\right) = -\frac{q}{R\theta_0^2}\sum_{j=1}^{N} B_{ij}\, u_j$$

$$\frac{EI}{R^2}\left(-R\beta_i + \frac{1}{\theta_0^2}\sum_{j=1}^{N} B_{ij}\, u_j\right) + \frac{GJ}{R^2\theta_0^2}\left(\sum_{j=1}^{N} B_{ij}\, u_j + R\sum_{j=1}^{N} B_{ij}\,\beta_j\right) = 0$$

.... (19)

The boundary conditions for clamped ends, given by equations (9), can be expressed in differential quadrature form as follows :

$$\beta_1 = 0 \quad at \quad X = 0$$
$$\beta_N = 0 \quad at \quad X = 1$$
$$u_1 = 0 \quad at \quad X = 0$$
$$u_N = 0 \quad at \quad X = 1$$

$$\sum_{j=1}^{N} A_{2j}\, u_j = 0 \quad at \quad X = 0 + \delta$$
$$\sum_{j=1}^{N} A_{(N-1),}\, u_j = 0 \quad at \quad X = 1 - \delta$$

.... (20)

The boundary conditions for simply supported ends given by equations (10) can be expressed in differential quadrature form as follows :

$$\beta_1 = 0 \quad at \quad X = 0$$
$$\beta_N = 0 \quad at \quad X = 1$$
$$u_1 = 0 \quad at \quad X = 0$$
$$u_N = 0 \quad at \quad X = 1$$

$$\sum_{j=1}^{N} B_{2j}\, u_j = 0 \quad at \quad X = 0 + \delta$$
$$\sum_{j=1}^{N} B_{(N-1),}\, u_j = 0 \quad at \quad X = 1 - \delta$$

.... (21)

Similarly, the boundary conditions for one clamped and one simply supported end, given by equations (11), can be expressed in differential quadrature form as

$$\beta_1 = 0 \quad at \quad X = 0$$
$$\beta_N = 0 \quad at \quad X = 1$$
$$u_1 = 0 \quad at \quad X = 0$$
$$u_N = 0 \quad at \quad X = 1$$

$$\sum_{j=1}^{N} A_{2j}\, u_j = 0 \quad at \quad X = 0 + \delta$$
$$\sum_{j=1}^{N} B_{(N-1),}\, u_j = 0 \quad at \quad X = 1 - \delta$$

.... (22)

This set of equations together with the appropriate boundary conditions can be solved for the out-of-plane buckling.

5. Numerical Results and Comparisons

In-plane and out-of-plane buckling parameters $q^*(-qR^3/EI)$ are calculated by the differential quadrature method and are presented together with existing exact solutions by Dinik ; see Timoshenko and Gere[6]. The critical values q^* are evafuated for the case of various end conditions and opening angles.

Table 1 presents the results of convergence studies relative to the number of grid points N with $\theta_0=180°$. The data show that the accuracy of the numerical solution increases with increasing N. Then numerical instabilities arise if N becomes too large (possibly greater than approx. 15). Table 2 shows the sensitivity of the solution to the choice of δ for the case of both ends simply supported. The optimal value for δ is found to be 1×10^{-6}, which is obtained from trial-and-error calculations. The solution accuracy decrcases due to numerical instabilities if δ becomes too big (possibly greater than approx. 1×10^{-2}).

Table 1 Critical value of in-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends simply supported for a range of grid points N; $\delta=1\times10^{-6}$

Timoshenko and Gere[6] θ_0, degrees	Number of grid points			
180°	9	11	13	15
3.0	2.989	2.999	2.986	2.887

Table 2 Critical value of in-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends simply supported for a range of δ; $N=11$

Timoshenko and Gere[6] θ_0, degrees	δ			
180°	1×10^{-3}	1×10^{-4}	1×10^{-5}	1×10^{-6}
3.0	3.015	2.986	2.985	2.999

In Tables 3 and 4, the critical pressures of in-plane buckling determined by the DQM are compared with the exact solution for the case of clamped and simply supported ends. Table 5 shows the numerical results by the differential quadrature method for the case of clamped-simply supported ends without comparison since no data are avallable.

Table 3 Critical value of in-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends clamped; $N=11$ and $\delta=1\times10^{-6}$

θ_0, degrees	$q^*=q_{cr}R^3/EI$	
	Timoshenko and Gere[6]	DQM
30°	294	294.6
60°	73.3	73.38
90°	32.4	32.44
120°	18.1	18.17
150°	11.5	11.58
180°	8	7.970

Table 4 Critical value of in-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends simply supported; $N=11$ and $\delta=1\times10^{-6}$

θ_0, degrees	$q^*=q_{cr}R^3/EI$	
	Timoshenko and Gere[6]	DQM
30°	143	142.6
60°	35	34.99
90°	15	15.0
120°	8	8.011
150°	4.76	4.758
180°	3.0	2.999

Table 5 Critical value of in-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with clamped-simply supported ends; $N=11$ and $\delta=1\times10^{-6}$

θ_0, degrees	$q^*=q_{cr}R^3/EI$
	DQM
30°	205.0
60°	50.72
90°	22.15
120°	12.14
150°	7.537
180°	5.065

In Tables 6 and 7, the critical pressures of twist-buckling by the DQM are compared with the exact solution for the case of simply supported ends with the stiffness parameters $k(=GJ/EI)$. Tables 8 and 9 show the results by the differential quadrature method for the case of clamped and clamped-simply supported

Table 6 Critical value of out-of-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends simply supported ; N=11 and $\delta=1\times10^9$ and k=0.5

θ_0, degrees	$q^*=q_{cr}R^3/EI$	
	Timoshenko and Gere[b)]	DQM
30°	32.24	32.23
60°	5.818	5.817
90°	1.50	1.50
120°	0.3667	0.3673
150°		0.05612
180°		0.0

Table 7 Critical value of out-of-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends simply supported ; N=11 and $\delta=1\times10^9$ and k=1.0

θ_0, degrees	$q^*=q_{cr}R^3/EI$	
	Timoshenko and Gere[b)]	DQM
30°	33.11	33.11
60°	6.40	6.40
90°	1.80	1.80
120°	0.4808	0.4806
150°		0.0792
180°		0.0

ends without comparison since no data are also available. Table 10 presents the results with various stiffness parameters k for the case of both ends simply supported. From Tables 3~9, it is seen that the critical loads of the member with clamped ends are much higher than those of the member with simply supported ends and those of the member with mixed clamped-simply supported ends. The critical pressure can be increased by decreasing the opening angle θ_0. From Table 10, the critical value of out-of-plane buckling increases as the stiffness parameter increases. As can be seen, the numerical results by the differential quadrature method show good to excellent agreement with the exact solutions.

6. Conclusions

The differential quadrature method was applied to the computation of the eigenvalues of the equations governing in-plane buckling and twist-buckling under uniform pressure. The present approach gives excellent results for the cases treated while requiring only a limited number of grid points ; only eleven discrete points were used for the evaluation.

Table 8 Critical value of out-of-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends clamped and clamped-simply supported ; N=11, $\delta=1\times10^{-9}$ and k=0.5

θ_0, degrees	$q^*=q_{cr}R^3/EI$ (DQM)	
	clamped-clamped	clamped-simply supported
30°	140.2	69.84
60°	32.61	15.08
90°	13.14	5.428
120°	6.621	2.378
150°	3.780	1.160
180°	2.338	0.610

Table 9 Critical value of out-of-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends clamped and clamped-simply supported ; N=11, $\delta=1\times10^{-9}$ and k=1.0

θ_0, degrees	$q^*=q_{cr}R^3/EI$ (DQM)	
	clamped-clamped	clamped-simply supported
30°	141.1	70.74
60°	33.30	15.75
90°	13.60	5.841
120°	6.932	2.620
150°	4.006	1.306
180°	2.525	0.7099

Table 10 Critical value of out-of-plane buckling parameter, $q^*=q_{cr}R^3/EI$, with both ends simply supported ; N=11, $\delta=1\times10^{-9}$ and θ_0=90°

k, (GJ/EI)	$q^*=q_{cr}R^3/EI$	
	Timoshenko and Gere[b)]	DQM
0.005		0.0442
0.2		0.9999
0.5	1.5	1.50
1.0	1.8	1.80
1.5		1.929
1.625	1.950	1.950

New results are given for two sets of boundary conditions not considered by previous investigators for out-of-plane buckling : clamped-clamped and clamped-simply supported ends.

The study reported herein is sponsored by Hoseo University research funds, the latter half of 1997.

REFERENCES

1) M. Ojalvo, E. Demuts and F. Tokarz, "Out-of-Plane Buckling of Curved Members", J. Struct. Dvi., ASCE, Vol. 95, pp. 2305~2316, 1969.
2) V.Z. Vlasov, Thin Walled Elastic Beams, 2nd edn, English Translation, National Science Foundation, Washington, D.C., 1961.
3) J.A. Cheney, "Bending and Buckling of Thin-Walled Open-Section Rings", J. Engng Mech. Div., ASCE, Vol. 89, pp. 17~44, 1963.
4) C.H. Yoo and P.A. Pfeiffer, "Elastic Stability of Curved Members", J. Struct, Engng, ASCE, Vol. 109, pp. 2922~2940, 1983.
5) J.P. Papangelis and N.S. Trahair, "Flexural-Torsional Buckling of Arches", J. Struct. Engng, ASCE, Vol. 113, pp. 889~906, 1987.
6) S.P. Timoshenko and J.M. Gere, Theory of Elastic Stability, 2nd edn, McGraw-Hill, New York, 1961.
7) N.S. Trahair and J.P. Papangelis, "Flexural-Torsional Buckling of Monosymmetric Arches", J. Struct. Engng, ASCE, Vol. 113, pp. 2271~2288, 1987.
8) Y.B. Yang and S.R. Kuo, "Static Stability of Curved Thin-Walled Beams", J. Struct. Engng, ASCE, Vol. 112, pp. 821~841, 1986.
9) S.R. Kuo and Y.B. Yang, "New Theory on Buckling of Curved Beams", J. Engng Mech., ASCE, Vol. 117, pp. 1698~1717, 1991.
10) Y.J. Kang and C.H. Yoo, "Thin-Walled Curved Beams, II : Analytical Solutions for Buckling of Arches", J. Struct. Engng, ASCE, Vol. 120, pp. 2102~2125, 1994.
11) Y.L. Pi, J.P. Papangelis and N.S. Trahair, "Prebuckling Deformations and Flexural-Torsional Buckling of Arches", J. Struct. Engng, ASCE, Vol. 121, pp. 1313~1322, 1995.
12) K. Kang and C.W. Bert, "Flexural-Torsional Buckling Analysis of Arches with Warping Using DQM", Engng Struct., Vol. 19, pp. 247~254, 1997.
13) R.E. Bellman and J. Casti, "Differential Quadrature and Long-Term Integration", J. Math, Anal. Applic., Vol. 34, pp. 235~238, 1971.
14) A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, 4th edn, Dover, New York, 1944.
15) T. Wah, "Buckling of Thin Circular Rings under Uniform Pressure", Int. J. Solids Struct., Vol. 3, pp. 967~974, 1967.
16) S.K. Jang, C.W. Bert and A.G. Striz, "Application of Differential Quadrature to Static Analysis of Structural Components", Int. J. Numer. Mech. Engng, Vol. 28, pp. 561~577, 1989.
17) A.R. Kukreti, J. Farsa and C.W. Bert, "Fundamental Frequency Tapered Plates by Differential Quadrature", J. Engng Mech., ASCE, Vol. 118, pp. 1221~1238, 1992.
18) J. Farsa, A.R. Kukreti and C.W. Bert, "Fundamental Frequency of Laminated Rectangular Plates by Differential Quadrature Method", Int. J. Numer. Engng, Vol. 36, pp. 2341~2356, 1993.
19) M. Malik and C.W. Bert, "Differential

Quadrature Solution for Steady State Incompressible and Compressible Lubrication Problems", J. Tribology, ASME, Vol. 116, pp. 296~302, 1994.

20) R.W. Hamming, Numerical Methods for Scientists and Engineers, 2nd edn, McGraw-Hill, New York, 1973.