J. Korea Soc. Math. Ed. Series B: The Pure and Applied Mathematics
4, No. 2 (1997) pp. 137–144

ON THE OPTIMAL COVERING OF EQUAL METRIC BALLS IN A SPHERE

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ABSTRACT. In this paper we consider covering problems in spherical geometry. Let $\mathrm{cov}_q S_1^n$ be the smallest radius of q equal metric balls that cover n-dimensional unit sphere S_1^n. We show that $\mathrm{cov}_q S_1^n = \frac{\pi}{2}$ for $2 \le q \le n+1$ and $\pi - \arccos(\frac{-1}{n+1})$ for $q = n+2$. The configuration of centers of balls realizing $\mathrm{cov}_q S_1^n$ are established, simultaneously. Moreover, some properties of $\mathrm{cov}_q X$ for the compact metric space X ,in general, are proved.

1. Introduction

Packing (or Covering) problem has been an exciting topic throughout the history of geometry. In its most general form, it asks for the optimal distribution of the sets $X_1, X_2, \cdots$ into a given space X. The classical problem is to find the most 'efficient' packing of $(n-1)-$dimensional sphere in the n-dimensional Euclidean space. It leads an outstanding, natural problem in non-euclidean category. Here we are only interested in the following basic problem for the spherical geometry.

Covering (Packing)Problem on S^n. : *What is the smallest(largest) radius r_q of q equal metric balls that can cover(pack) S^n? How must the balls be arranged to achieve this minimum(maximum), and when is there an essentially unique arragement?*

Fairly recently, Grove and Wilhelm solved optimal packing problems on S^n and derived some sphere theorems using them in [4].

Received by the editors Sep. 8, 1997 and, in revised form Dec. 29, 1997.
1991 *Mathematics Subject Classification.* primary 52C17.
Key words and phrases. Spherical Geometry, Covering Radii, Covering Problems.
Research supported in part by KRF 1997-003-D00014

Recall that the q-th packing raius of a compact metric space X is defined by

$$2\text{pack}_q X = \max_{(p_1,\cdots,p_q)} \min_{i<j} d(p_i, p_j),$$

where the maximum is taken over all configurations of q distinct points in X, and the minimum is taken over all pairwise distances in any such a configuration. Then, the followings are true :

Theorem(G-W). *Let S_1^n be an n-dimensional unit sphere with $n \geq 2$,then*

(i) for $2 \leq q \leq n+2$,

$$pack_q S_1^n = pack_q S_1^{q-2} = \frac{1}{2}\arccos(\frac{-1}{q-1})$$

and the configuration of points realizing $pack_q S_1^n$ is uniquely determined as the vertices of a regular, inscribed $(q-1)$-simplex in some totally geodesic $S_1^{q-2} \subset S_1^n \subset R^{n+1}$.

(ii) for $n+3 \leq q \leq 2n+2$,

$$pack_q S_1^n = \frac{\pi}{4}$$

and the configuration of points realizing $pack_q S_1^n$ is uniquely determined as the $(\pm e_1, \pm e_2, \cdots, \pm e_{n+1})$, where $(e_1, e_2, \cdots, e_{n+1})$ is an orthonomal basis for R^{n+1}.

The covering problem on S^n is motivated by the above result. Packing and covering usually go side-by-side. There are dual covering problems to many packing problems. Similar techniques may sometimes be used in both packing and covering situations, even though we will use quite different method in this case.

In general, problems involving packing or covering with open or closed balls-respectively-are often expressed in terms of arranging points so that they are not too close or too far from each other. But it is an exceedingly hard problem to determine the value and distribution of points realizing $\text{pack}_q X$ as well as $\text{cov}_q X$(See section 2 for the definition). It is still unsolved even for $X = S_1^2$. $\text{pack}_q S_1^2$ and the corresponding arrangment of (open) balls have been found for $q \leq 12$ and $q = 24$. The exact solutions of $\text{cov}_q S_1^2$ is known only for $q \leq 7$ and $q = 10, 12$ and 14. For other values of q, various upper and lower bounds for $\text{pack}_q S_1^2$ (or $\text{cov}_q S_1^2$) have been obtained, but many of these could certainly be improved. (See [2,Chapter D].)

The paper is organized as follows. In section 2 we define q-th covering radius of a metric space and present some properties of covering radii. Section 3 is devoted to the proof of our main theorem. Some spherical geometric lemmas are discussed and the covering theorem is followed.

For basic results and concepts from spherical geometry and packing(or covering) problems that will be used freely, we refer to [2] and [5].

2. Covering radii of a compact metric space

Recall that the radius of a compact metric space X is defined as $\text{rad}X = \min_{p\in X}\max_{q\in X} d(p,q)$,which is related with the diameter by

$$\text{rad}X \leq \text{diam}X \leq 2\text{rad}X,$$

where the last inequality is strict for closed Riemannian manifolds. Clearly, $\text{rad}X$ is the smallest positive number r so that X can be covered by the closed ball $\overline{B}(p,r) = \{x \in X | d(p,x) \leq r\}$ for some $p \in X$.

As we described in the introduction, we can define a non-increasing sequence, so-called, covering radii as the dual notion of packing radii as follows :

Definition 2.1. Let X be a metric space and $\overline{B}(p,r)$ be a closed r-ball centered at p in X. Then we call $r_q(p_1,p_2,\cdots,p_q)$ the q-th covering radius of X with respect to $\{p_1,p_2,\cdots,p_q\}$,where $r_q\colon X^q \to R^+ \cup \{0\}$ is the smallest number satisfying

$$\overline{B}(p_1,r) \cup \overline{B}(p_2,r) \cup \cdots \cup \overline{B}(p_q,r) = X$$

for any q-tuple $(p_1,p_2,\cdots,p_q) \in X^q$. The q-th covering radius of X is defined by

$$\text{cov}_q X = \min_{p_1,\cdots,p_q} r_q(p_1,p_2,\cdots,p_q)$$

In other words, X is covered by q (overlapping) closed balls of radius $\text{cov}_q X$, and $\text{cov}_q X$ is the smallest number satisfying this property. Moreover,we can view $\text{rad}X$ as the first term in a non-increasing sequence of covering radii, i.e.,

$$\text{rad}X = \text{cov}_1 X \geq \text{cov}_2 X \geq \cdots \geq \text{cov}_q X \geq \cdots \geq 0.$$

The q-th packing radius is nothing but one half of the minimal distance between pairs of points, if we know a configuration of points which realize an optimal packing. But the situation is more complicated in the covering radii case. This is due to the fact that an overlapping of metric balls is allowed in the covering problem. However, the above geometric definition can be explicitly expressed as the following for the compact metric space.

Theorem 2.2. *Let X be a compact metric space, then*

$$cov_q X = \min_{p_1,\cdots,p_q} d_H(X,\{p_1,p_2,\cdots,p_q\}) = \min_{p_1,\cdots,p_q} \max_{x\in X} \min_i d(x,p_i),$$

where $p_i \in X, i = 1,2,\cdots,q$, and d_H is the classical Hausdorff distance.

Proof. Fix $(p_1,p_2,\cdots,p_q) \in X^q$. Since X is covered by q r_q-balls, for any $x \in X$ there exist $j, 1 \leq j \leq q$, such that

$$d(x,p_j) \leq r_q = r_q(p_1,p_2,\cdots,p_q).$$

Hence, clearly we have

$$\max_{x\in X} \min_i d(x,p_i) \leq r_q \qquad (*).$$

On the other hand, let $s = \max_{x\in X} \min_i d(x,p_i)$. Then, for any $x \in X$,

$$\min_i d(x,p_i) \leq \max_{x\in X} \min_i d(x,p_i) = s,$$

which means $\overline{B}(p_1,s) \cup \overline{B}(p_2,s) \cup \cdots \cup \overline{B}(p_q,s) = X$. Since r_q is the smallest radius of q balls satisfying the above property,

$$\max_{x\in X} \min_i d(x,p_i) = s \geq r_q \qquad (**).$$

By combining $(*)$ and $(**)$, $r_q(p_1,p_2,\cdots,p_q) = \max_{x\in X} \min_i d(x,p_i)$. But, by the definition of the classical Hausdorff distance,

$$d_H(X,\{p_1,p_2,\cdots,p_q\}) = \max_{x\in X} d(x,\{p_1,p_2,\cdots,p_q\}) = \max_{x\in X} \min_i d(x,p_i) = r_q \quad \square.$$

As a metric invariant, the definition of covering radii leads that $\text{cov}_q X$ is a continuous function on the space of compact metric spaces relative to the Gromov-Hausdorff topology. We refer to the celebrated [3] for the Gromov-Hausdorff distance.

Theorem 2.3. *$cov_q X$ is continuous relative to the Gromov-Hausdorff topology.*

Proof. Let X and Y be compact metric spaces. Given $\epsilon > 0$, choose $\delta > 0$ such that $\delta < \frac{\epsilon}{2}$. Suppose $d_{G-H}(X,Y) < \delta$, then there is, by definition of the Gromov-Hausdorff distance, a metric d on $X \coprod Y$ extending the metrics on X and Y so that the classical Hausdorff distance between X and Y in $X \coprod Y$ is less than or equal to ϵ. Moreover, given $p_i, x \in X$ there are $u_i, y \in Y$ such that $d(p_i, u_i) \leq \delta$ and $d(x,y) \leq \delta$. By the triangle inequality,

$$d(p_i, x) \leq d(p_i, u_i) + d(u_i, x) \leq d(p_i, u_i) + d(u_i, y) + d(x, y),$$

Hence $d(p_i, x) \leq 2\delta + d(u_i, y)$ and consequently, $\mathrm{cov}_q X \leq 2\delta + \mathrm{cov}_q Y$. By the symmety, we also get $\mathrm{cov}_q Y \leq 2\delta + \mathrm{cov}_q X$. Therefore,

$$|\mathrm{cov}_q X - \mathrm{cov}_q Y| \leq 2\delta < \epsilon. \qquad \square$$

3. Covering radius of a sphere up to $n+2$

Let $P_0, P_1, \cdots, P_{n+1}$ be vertices of a regular inscribed $(n+1)$-simplex in S_1^n. Since $P_1 + P_2 + \cdots + P_{n+1} = 0$ as vectors in R^{n+1}, the spherical distance between each pair of vertices is equal to $\arccos(\frac{-1}{n+1})$ by applying the inner product. For notational convinience, let $\ell_n = \arccos(\frac{-1}{n+1})$.

From now on, we devote the rest of this section to compute the value of $\mathrm{cov}_q S_1^n$ up to $q = n+2$ and, more importantly in the geometrical sense, determine the configuration of q points realizing $\mathrm{cov}_q S_1^n$. In [4], they used the metric argument for the determination of distribution of points realizing $\mathrm{pack}_q S_1^n$. But the volume argument seems to be much easier to treat for the covering radii case than the metric one. We need the following lemma from [1] to use our method.

Lemma 3.1 (Bőröczky). *Let Δ be a (spherical) simplex in a closed ball $\overline{B}$ in S_1^n and Δ_0 be an inscribed regular simplex of $\overline{B}$, then the volumes $vol(\Delta)$, $vol(\Delta_0)$ of the simplexes satisfy*

$$vol(\Delta) \leq vol(\Delta_0)$$

and equality is attained if and only if Δ and Δ_0 are congruent.

Now we will return to our main concern - determination of $\mathrm{cov}_q S_1^n$ up to $q = n+2$. It can be easily checked that $\mathrm{cov}_q S_1^1 = \frac{\pi}{q}$ for the unit circle. For $n \geq 2$, it is clear that $\mathrm{cov}_2 S_1^n = \frac{\pi}{2}$ and P_2 is the antipodal point of P_1 if P_1 and P_2 realize $\mathrm{cov}_2 S_1^n$.

Now consider the case for $q = n+1$. One can observe that any distribution of $n+1$ points in S_1^n is restricted, i.e., they must stay in some n-dimensional affine subspace of R^{n+1} and hence in S_k^{n-1}, a sphere with radius $= \frac{1}{\sqrt{k}}$, which is contained in S_1^n, for some $k \geq 1$. So consider the south and north pole of that subspace, then $\mathrm{cov}_{n+1} S_1^n$ should be at least $\frac{\pi}{2}$ in order to cover both poles. By the non-increasingness of cov_q we can get

$$\frac{\pi}{2} = \mathrm{cov}_2 S_1^n \geq \mathrm{cov}_3 S_1^n \geq \cdots \geq \mathrm{cov}_{n+1} S_1^n \geq \frac{\pi}{2}.$$

Moreover, clearly we can not determine uniquely the configuration of points realizing $\mathrm{cov}_q S_1^n$ for $3 \leq q \leq n+1$. So we have proved the first part of the following theorem.

Theorem 3.2. *For $n \geq 2$,*

$$\mathrm{cov}_q S_1^n = \begin{cases} \pi & q = 1, \\ \frac{\pi}{2} & 2 \leq q \leq n+1, \\ \pi - \arccos(\frac{-1}{n+1}) & q = n+2. \end{cases}$$

and the configuration of points realizing $\mathrm{cov}_{n+2} S_1^n$ is uniquely determined as the vertices of a regular, inscribed $(n+1)$-simplex in S_1^n.

Proof. First let $P_0, P_1, \cdots, P_{n+1}$ be vertices of a regular inscribed $(n+1)$-simplex in S_1^n. Let $d(P_i, P_j) = \ell_n$ for $i \neq j$ and fix P_0. Then we have $d(-P_0, P_i) = \pi - \ell_n$ for $i = 1, 2, \cdots, n+1$.

Since $P_0, P_1, \cdots, P_{n+1}$ are contained in the (closed)$(\pi - \ell_n)$-ball centered at $-P_0$, the common radius of $n+1$ balls centered at $P_1, P_2, \cdots, P_{n+1}$ should be at least $\pi - \ell_n$ in order to cover the (spherical) n-simplex $< P_1 P_2 \cdots P_n >$.

Since P_0 is arbitrary and $\bigcup_{i=0}^{n+1} < P_0 P_1 \cdots \widehat{P_i} \cdots P_n > = S_1^n$, $\pi - \ell_n$ is the $(n+2)$-th covering radius with respect to given $\{P_0, P_1, \cdots, P_{n+1}\}$. So, by definition, we have

$$\mathrm{cov}_{n+2} S_1^n \leq \pi - \ell_n \qquad (A)$$

Now suppose $\{Q_0, Q_1, \cdots, Q_{n+1}\}$ is a configuration of points in S_1^n which realize $\mathrm{cov}_{n+2}S_1^n$. Choose any $n+1$ points from $\{Q_0, Q_1, \cdots, Q_{n+1}\}$, then they lie on S_k^{n-1} for some $k > 1$. If , for $0 \le i \le n+1$,

$$\dim(\mathrm{conv}(Q_0, Q_1, \cdots, \widehat{Q_i}, \cdots, Q_{n+1})) < n$$

i.e., they lie on some m-dimensional subspace for $m \le n-1$, then $Q_0, Q_1, \cdots, Q_{n+1}$ lie on $S_{k'}^{m-1}$ for $k' > 1$. It means $\{Q_0, Q_1, \cdots, Q_{n+1}\}$ cannot realize $\mathrm{cov}_{n+2}S_1^n$. Let $\Delta^i = < Q_0 Q_1 \cdots \widehat{Q_i} \cdots Q_{n+1} >$ be a (spherical) n-simplex in S_1^n. Then

$$\sum_{i=0}^{n+1} \mathrm{vol}(\Delta^i) = \mathrm{vol}(S_1^n) = (n+2)\mathrm{vol}(\Delta_0), \tag{$*$}$$

where Δ_0 is a regular (spherical) n-simplex inscribed in the ball with radius$= \pi - \ell_n$.

It is immediate from $(*)$ that there is Δ^j having the largest volume and satisfying

$$\mathrm{vol}(\Delta^j) \ge \mathrm{vol}(\Delta_0) \tag{$**$}$$

Let B_j be the ball so that Δ^j is inscribed in B_j, then the radius of B_j is greater than or equal to $\pi - \ell_n$ by Lemma 3.1. So we have

$$\mathrm{cov}_{n+2}S_1^n \ge \pi - \ell_n \tag{B}$$

By combining (A) and (B), hence we get $\mathrm{cov}_{n+2}S_1^n = \pi - \ell_n$.

The above fact implies the radius of $B_j = \pi - \ell_n$, which means Δ^j is regular and $\mathrm{vol}(\Delta^j) = \mathrm{vol}(\Delta_0)$ by $(**)$ and lemma 3.1 again.

Since Δ^j realize the largest volume, we have the following fact from $(*)$:

$$\mathrm{vol}(\Delta^0) = \mathrm{vol}(\Delta^1) = \cdots = \mathrm{vol}(\Delta^{n+1}) = \mathrm{vol}(\Delta_0)$$

Therefore, $\{Q_0, Q_1, \cdots, Q_{n+1}\}$ are vertices of a regular inscribed $(n+1)$-simplex in S_1^n. □

REFERENCES

1. K. Bőrőczky, *On the extremum property of the regular simplex in S^d*, Intuitive Geometry (K. Bőrőczky and L. Tóth, eds.), Colloquia Mathematica Societatis János Bolyai 58, 1985, pp. 117–121.
2. H. Croft, K. Falconer, and R. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, Berlin, 1991.

3. M. Gromov, J. Lafontaine, and P. Pansu, *Structures métriques pour les variétés riemanniennes*, Cedic/Fernand Nathan, 1981.
4. K. Grove and F. Wilhelm, *Hard and soft packing radius theorems*, Ann. of Math. **142** (1995), 213–237.
5. W. Y. Hsiang, *Lecture Notes on Classical Geometry of Spheres*, Berkeley Lecture Notes.

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