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ASYMPTOTIC VALUES OF MEROMORHPIC FUNCTIONS WITHOUT KOEBE ARCS

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ABSTRACT. A simple proof for the special case of the McMillan and Pommerenke Theorem on the asymptotic values of meromorphic functions without Koebe arcs is derived from the author's result on the boundary behavior of meromorphic functions without Koebe arcs.

In [1] McMillan and Pommerenke have proved that if $f(z)$ is an analytic function in the unit disc, without Koebe arcs, then for each point t on the unit circle, either $f(z)$ has an asymptotic value at t, or every neighborhood of t contains ternary Gross domains of $f(z)$, and furthermore $r(t,d) \to 0$ as $d \to 0$, where $r(t,d)$ denotes the supremum of the Euclidean diameters of the Gross domains of $f(z)$ intersecting $|z-t| < d$.

In this note we define a Gross domain on the Riemann sphere of meromorphic functions and consider the properties of meromorphic functions without Koebe Arcs.

Let D be a domain on the Riemann sphere S, and let P be a point in D, Q a point in S. By a suitable linear transformation L the point P may be transformed to the south pole, the point Q to the north pole. Then D is said to be (P,Q)-star-shaped if the stereographic projection of $L(D)$ is star-shaped with respect to the origin in the complex plane.

A domain D is said to be star-shaped if it is (P,Q)-star-shaped for some points P in D and Q in S.

A Gross domain G of a meromorphic function on D is defined to be a subdomain of D having the following properties :

(a) $f(z)$ maps G one-to-one onto a star-shaped domain on the Riemann sphere and

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(b) G is not properly contained in any other subdomain of D having the property (a).

In [2] we have shown the following theorems.

Theorem 1. *Let $f(z)$ be a meromorphic function in the unit disc, without Koebe arcs, which has asymptotic values on a dense set in an arc on the unit circle C. Then for each point t on the circle, either*

(a) $f(z)$ has an asymptotic value at t, or

(b) every neighborhood of t contains non-degenerate Gross domains of $f(z)$, and furthermore $r(t,d) \to 0$, as $d \to 0$ where $r(t,d)$ denotes the supremum of the Euclidean diameters of the Gross domains of $f(z)$ intersecting $\{z||z-t| < d\}$.

Theorem 2. *Let $f(z)$ be a meromorphic function in the unit disc, without Koebe arcs, and assume that $f(z) = h(z)/g(z)$, where $h(z)$ is an analytic function and $g(z)$ is a bounded analytic function which is not identically equal to zero. Then for each t on the unit circle, either*

(a) $f(z)$ has an asymptotic value at t, or

(b) every neighborhood of t contains non-degenerate Gross domains of $f(z)$, and furthermore $r(t,d) \to 0$ as $d \to 0$.

In [1] McMillan and Pommerenke have proved the following theorems. We prove them as corollaries of theorem 2.

Theorem 3. *Let $f(z)$ be an analytic function in the unit disc, without Koebe arcs. Then for each t on the unit circle, either*

(a) $f(z)$ has an asymptotic value at t, or

(b) every neighborhood of t contains ternary Gross domains of $f(z)$, and furthermore $r(t,d) \to 0$ as $d \to 0$.

Proof. In the definition of the (P,Q)-star-shaped domain on the Riemann sphere we take the north pole for Q, and let $g \equiv 1$ in Theorem 2.

Theorem 4. *A locally univalent normal analytic function has three distinct asymptotic values on every arc A on the unit circle.*

Proof. Suppose that the assertion is false. Then there is an interior point on the arc A at which there is no asymptotic value. Otherwise the normal function would have the same asymptotic value, hence angular limit, on a set of positive measure.

Then, by Priwalow uniqueness theorem, $f(z)$ must be a constant, contrary to our assumption that $f(z)$ is locally univalent.

By Theorem 2, there exists a ternary domain G arbitrarily near t. Since $f(z)$ is locally univalent any ray of G is a Jordan arc in D except for its end point t on G where $f(z)$ has a finite or an infinite asymptotic value. Since G is ternary there are three rays of G ending at points of A. The asymptotic values on these rays are distinct because at least two of them are finite.

Remark. It has actually been proved by McMillan and Pommerenke for locally univalent meromorphic functions without Koebe arcs.

Using the new definition of the Gross domain, the following theorem can be proved.

Theorem 5. *Let $f(z)$ be a locally univalent meromorphic function without Koebe arcs. Then for each point t on the unit circle, either*

(a) $f(z)$ has an asymptotic value at t, or

(b) every neighborhood of t contains non-degenerate Gross domains of $f(z)$, and moreover $r(t,d) \to 0$ as $d \to 0$.

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