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THE STUDY OF THE DIFFERENCE BETWEEN GALLOPING CABLE AND SUSPENSION BRIDGE CABLE

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ABSTRACT. We consider the common and different results between the oscillation of galloping cable and the oscillation of suspension bridge cable through the long-term behavior. Numerical results are presented by using the second-order Runge-Kutta method under various initial conditions. There appeared to be nonlinear forms. Periodicity, symmetry, and longitudinality are differently appeared in two kinds of cables.

1. Introduction

Cable vibration problems are of considerable antiquity. In the case of the Aeolian harp of the Greeks, musical sounds on exposure to the wind due to the strumming effect of the shed vortices are given out. Pythagoras and his disciples were keenly interested in string vibrations and understood at least qualitatively the relations between pitch of the note and the tension, length, and mass of the string that produced it [6].

The basic laws governing taut string vibrations were found experimentally by Mersenne and stated by him in 1636. The idea that a given string has many modes of vibration and the associated concept of internal nodes were established by Noble and Pigott in 1676 [9].

In 1744, Euler derived the correct equations for the large vibrations of a string in a plane. He regarded the equations as the limit of those for a finite collection of beads joined by massless springs as the number of beads approach infinity while their total mass remains fixed. The motion of the system of beads is described by a finite system of ordinary differential equations [2].

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The main ingredients of the system we shall study will be (a) a cable suspended between two supports of equal elevation, (b) the gravitational load on the cable which keeps it in a state of tension near equilibrium, and (c) some additional external forcing term due to some extraneous physical situation.

These three ingredients would be present in the following two phenomena; the question of galloping cables and the question of the response to periodic forcing of cables in suspension bridge.

Galloping is the large amplitude up-and-down motion that can break lines and cause catastrophic failure towers of their supporting according to the Electric Power Research Institute[8].

In the case of galloping cables, the cables would be under the influence of additional gravitational forces due to ice formation, as well as periodic forces that might be either aerodynamic in origin, or might arise from the vibration of the supporting towers.

In the case of the suspension bridge cables, the cables are kept in an equilibrium position of extension by the suspended weight of the road-bed, and periodic forces would be exerted either by the aerodynamic effects of the wind on the road-bed or by the oscillations of the towers. The oscillation of the towers could be induced by motions of the side-spans, or wind effects on the towers, or wind effects on the cable system. These oscillations in the towers have been observed in the case of the Tacoma Narrows bridge to occur either in phase or out of phase[1].

The question we start to address in this paper is what happens when the amplitude of the motions, either through random gusts or other perturbations, escapes from the linear small-oscillation range.

"Linear thinking" would suggest that in the presence of damping (which is in all the system we treat), the cable ought eventually settle back down to the small oscillation after a suitable interval of transient behavior. However, there is a substantial body of work which suggests that as the system goes from a regime of extension to one of slackness, large-amplitude periodic solutions can exist, and even in the presence of a small forcing term and damping, a single random push can be sufficient to send the system from small oscillation to sustained large-amplitude and potentially destructive oscillation[3].

We present numerical results that a single random push can send the system from small oscillation to sustained large amplitude and potentially destructive oscillation.

Moreover, we discover that in the presence of solely vertical forcing, pronounced longitudinal motions can occur. Also, we observe the differences between the oscillation of the suspension bridge cable and oscillation of galloping cable.

2. Governing equations

We will treat a cable as a series of equally distributed point masses connected by nonlinear springs with the same unstretched lengths. To model the motion of the cable, the springs resist extension but not compression. The restoring forces are presumed to be proportional to the extension (but not compression) of a one-sided spring joining two particles. Usually, they are subject to vertical periodic forces. Damping force will be assumed to act in a direction opposite to the motion with a magnitude proportional to the instantaneous velocity.

We consider a cable which is hung between two fixed points at the same vertical level and distance L apart. Let the line joining two supports be the x-axis with two fixed points located at $x = 0$ and $x = L$. Let the instantaneous position of the i-th particle be $(x_i(t), y_i(t))$ at time t with the positive directions for x and y as shown in figure 1. Then Newton's second law and Hooke's law give rise to the following equations,

$$\begin{cases} \rho l d^2 x_i/dt^2 &= -k(\overline{P_{i-1}P_i} - l)^+ \cos\theta_i + k(\overline{P_i P_{i+1}} - l)^+ \cos\alpha_i - c l dx_i/dt \\ \rho l d^2 y_i/dt^2 &= -k(\overline{P_{i-1}P_i} - l)^+ \sin\theta_i + k(\overline{P_i P_{i+1}} - l)^+ \sin\alpha_i - c l dy_i/dt \\ & \quad +\rho l g + l f(\tilde{x}_i, t). \end{cases} \tag{1}$$

Here, $\tilde{x}_i = i\Delta x/L$, $\Delta x = L/(N+1)$, and N is the number of particles discretizing the cable. $\overline{P_{i-1}P_i}$ is the distance between P_{i-1} and P_i at time t and similarly for $\overline{P_i P_{i+1}}$, $1 \le i \le N$. In particular, $P_0(x_0, y_0) = P_0(0, 0)$ and $P_{N+1}(x_{N+1}, y_{N+1}) = P_{N+1}(L, 0)$ are the two stationary supports.

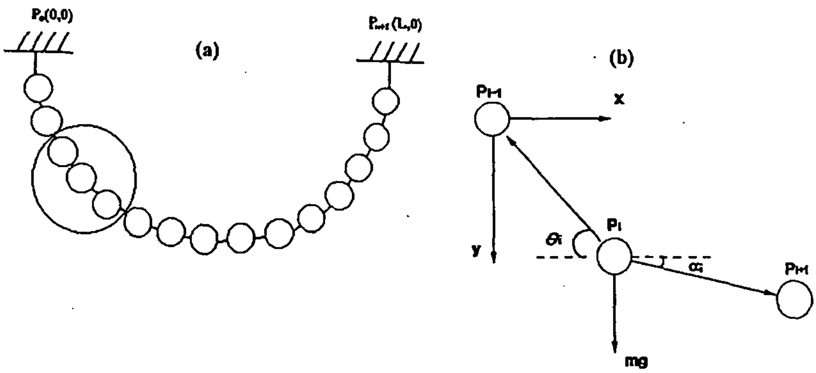

Figure 1: a) Discrete cable: Equally distributed particles are supported from two points by nonlinear springs with the same unstretched lengths. b) Expanded diagram of encircled part of a).

The term ρ denotes the mass per unit length of unstretched cable and the term l the unstretched length of the spring between two point masses. The spring constant is denoted by k and $k = \frac{EA}{l}$, where E is Young's modulus and A is the cross section area. As we double the number of particles in modeling a cable, k is increased by two. The damping coefficient per unit unstretched length is denoted by c and the accelation due to gravity by g. The angles that the cables make with the x-axis, which are shown in figure 1, are denoted by θ_i and α_i. Moreover, the notation u^+ is defined as

$$u^+ \equiv \begin{cases} u, & \text{if } u > 0. \\ 0, & \text{if } u \leq 0. \end{cases}$$

Such a nonlinear term comes from the fact that when the cables are stretched, there are restoring forces which are assumed to be proportional to the amount of stretching. However, when they are compressed, there is no restoring force exerted on them.

Substituting the geometric relations between angles and lengths into equations (1) and dividing by mass give the following equations,

$$\begin{aligned}\frac{d^2x_i}{dt^2} =& -\frac{k}{\rho l}(\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}-l)^+\frac{(x_i-x_{i-1})}{\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}}\\ &+\frac{k}{\rho l}(\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}-l)^+\frac{(x_{i+1}-x_i)}{\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}}\\ &-\frac{c}{\rho}dx_i/dt,\end{aligned}$$

$$\begin{aligned}\frac{d^2y_i}{dt^2} =& -\frac{k}{\rho l}(\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}-l)^+\frac{(y_i-y_{i-1})}{\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}}\\ &+\frac{k}{\rho l}(\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}-l)^+\frac{(y_{i+1}-y_i)}{\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}}\\ &-\frac{c}{\rho}dy_i/dt+g+\frac{1}{\rho}f(\tilde{x}_i,t).\end{aligned}$$

Here, $x_0(t)=0, x_{N+1}(t)=L, y_0(t)=0, y_{N+1}(t)=0, 1\le i\le N$.

The forcing term $f(\tilde{x}_i,t)$ is taken to be $\lambda\sin\mu t\sin\pi\tilde{x}_i$, which is the form of a standing wave.

Let $\vec{u}=(x_1,y_1,\cdots,x_N,y_N)^T, \vec{F}=(F_1,G_1,\cdots,F_N,G_N)^T$. Then the given non-autonomous system becomes

$$\frac{d^2\vec{u}}{dt^2}=\vec{F}(t,\vec{u}),$$

where

$$\begin{aligned}&F_i(x_{i-1},y_{i-1},x_i,y_i,x_{i+1},y_{i+1},t)\\ &=-\frac{k}{\rho l}(\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}-l)^+\frac{(x_i-x_{i-1})}{\sqrt{(x_i-x_{i-1})^2+(y_i-y_{i-1})^2}}\\ &+\frac{k}{\rho l}(\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}-l)^+\frac{(x_{i+1}-x_i)}{\sqrt{(x_{i+1}-x_i)^2+(y_{i+1}-y_i)^2}}\\ &-\frac{c}{\rho}dx_i/dt,\end{aligned}$$

$$
\begin{aligned}
&G_i(x_{i-1}, y_{i-1}, x_i, y_i, x_{i+1}, y_{i+1}, t) \\
&= -\frac{k}{\rho l}\left(\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} - l\right)^+ \frac{(y_i - y_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} \\
&+ \frac{k}{\rho l}\left(\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} - l\right)^+ \frac{(y_{i+1} - y_i)}{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}} \\
&- \frac{c}{\rho} dy_i/dt + g + \frac{1}{\rho} f(\tilde{x}_i, t).
\end{aligned}
$$

We will solve the system of differential equations numerically. Our method of solving this system is to use the 2nd-order Runge-Kutta method employing higher precision. In order to study the system numerically, we will search for stable periodic solutions by solving the initial value problem for various initial conditions and allowing the solution to run for large time.

3. Main Results

Based on the experience in [4][7], we expect to find interesting results near linear resonance. After finding linear resonance, we vary μ and λ under various initial conditions. The interesting solutions near linear resonance are investigated when the transient effects have been eliminated.

Since galloping cable is a continuous structure, we need more particles to approximate the cable. To approximate the cable, we solve the problem computationally for various different numbers of particles and we find that convergence is achieved, that is the motion stays the same even as we increase the number of particles. In fact, as the number of particles was increased to 63, 127, we got convergence for the solutions. We illustrate the 63 particle case for the galloping cable because the pictures for the larger number of particles are identical with figures presented.

On the other hand, since the suspension bridge has the vertical cables at regular intervals as shown in figure 2, the cable in the suspension bridge makes a more discrete model possible.

In discussing the motions of the cable of a suspension bridge, probably the more discrete case with fewer particles would be a better model than the more continuous case with more particles.

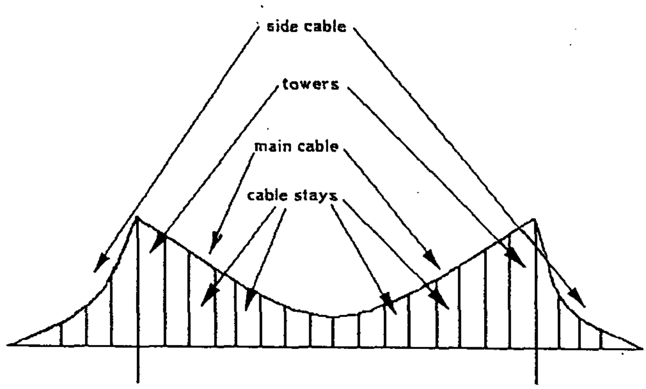

Figure 2: Diagram of the cable in the suspension bridge.

In all the numerical results, we let the distance L be 1, the total mass of the cable M be 5, the unstretched length of the cable ζ be 1.2, $c = \frac{1}{3}$, $EA = 19.2$, and $\rho = M/\zeta = 25/6$. When N is 63, $l = \zeta/(N+1) = 3/160$ and $k = \frac{EA}{l} = 1024$. But we let N be 15 for suspension bridge cable. Then $l = 1.2/16$ and $k = 256$.

We sample five solution profiles at an equally spaced time interval in one forcing period. In all figures except figure 4, the profiles in the last one period are shown.

We shall consider the response of the cables to forcing with no node(i. e. $f(\tilde{x}_i, t)$ $= \lambda \sin \mu t \sin \pi \tilde{x}_i$).

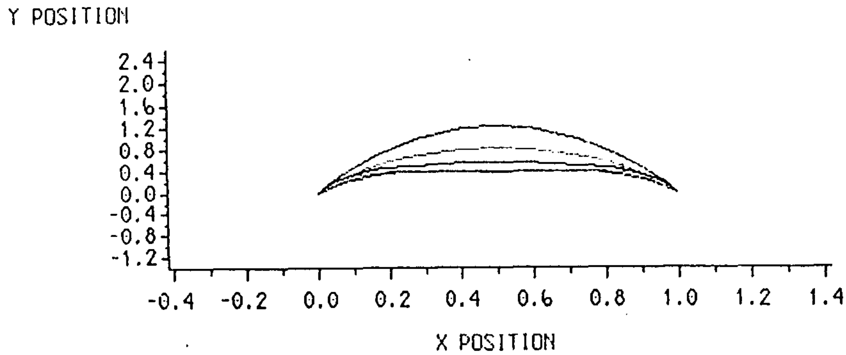

Figure 3: The large amplitude solution of suspension bridge cable model with $\mu =$ 5.2 and $\lambda = 0.2$.

3.1 Common Results

As long as the initial conditions remain small and there is a small periodic forcing after several hundred periods had elapsed, the cable has the reasonable figure which is a small amplitude solution. However, in the case of large initial condition, the long term solution of the initial value problem gives a large-amplitude solution.

Here, we can find some nonlinear forms. The first type of nonlinear form is a subharmonic solution. This is a solution with period double that of the forcing term. This is observed in figure 4.

The second type of nonlinear form is fuzzily periodic solution with a period approximate to that of the forcing term. This solution reflects the nodal character of the forcing term and occurs over a wider range of frequency and amplitude than the first type.

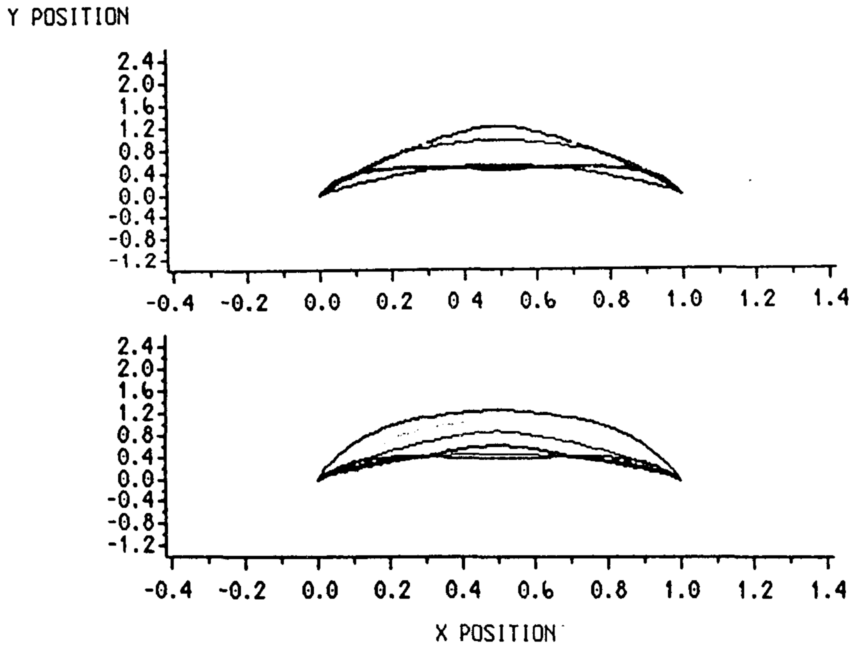

Figure 4: The first half and the second half in the last 2 periods of the large amplitude solution of galloping cable model (from top to bottom) are represented when $\mu = 5.2$ and $\lambda = 0.2$.

The third type of nonlinear form is a longitudinal motion. This is surprising, since purely vertical forcing can cause the cable to go into a large-amplitude vertical motion with a longitudinal component.

The second type and the third type are commonly appeared in galloping cable and in suspension bridge cable.

3. 2 Differences

In this section, we shall consider the differences between the oscillation of galloping cable and the oscillation of suspension bridge cable. The large number of experiments conducted allow us to get the following three differences between two kinds of cables. The following differences are, of course, appeared in large-amplitude solutions.

3.2.1 Periodic Solutions

Many of the 15-particle cases have periodic solutions with the same period as the forcing term while the 63-particle cases have subharmonic periodic solutions with period double that of the forcing term. Figure 3 of the 15 particle case shows a large amplitude solution at $\mu = 5.2$ and $\lambda = 0.2$. This solution is found to be as perfectly periodic as one can verify by numerical experiments. Figure 4 of the 63 particle case shows a large amplitude solution with period double that of the forcing term. Of course, figure 4 has the same parameters with figure 3.

In both cases, small amplitude solution exists with the same parameters. Figure 5 shows this solution. The same behavior is observed computationally over a wide range of frequencies and amplitudes. Hence, it is noted that multiple periodic solutions exist for galloping cable and suspension bridge cable.

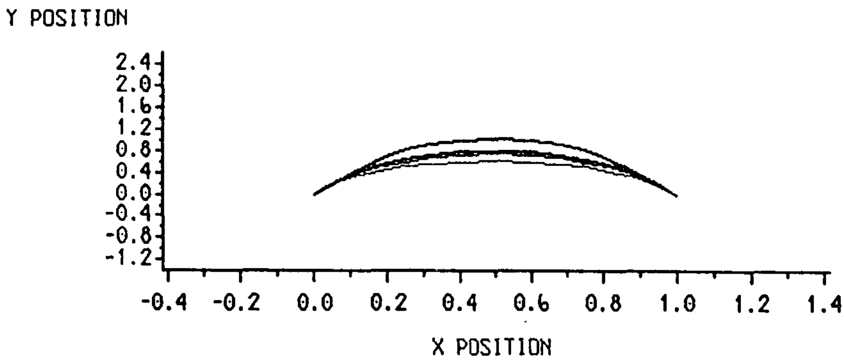

Figure 5: The small amplitude solution of both cable models with $\mu = 5.2$ and $\lambda = 0.2$. Here, we note that we get convergence for small amplitude solution.

3.2.2 Symmetric Solutions

Figures of fewer particle cases show more unsymmetric solutions than those of more particle cases. Figure 6 shows the solution when $\mu = 4.8$ and $\lambda = 0.6$. Figure 6 shows that the 15-particle case is more unsymmetric about the mid point than 63-particle case.

They are large-amplitude periodic solutions with no node. They are not perfectly periodic but have the same approximate period as the forcing term.

3.2.3 Longitudinal Solutions

The third difference appeared in the large-amplitude solutions is the longitudinality.

This phenomenon usually appears with a near-periodic solution, somewhat disorgarnized solution, which is of the same approximate period as the forcing term.

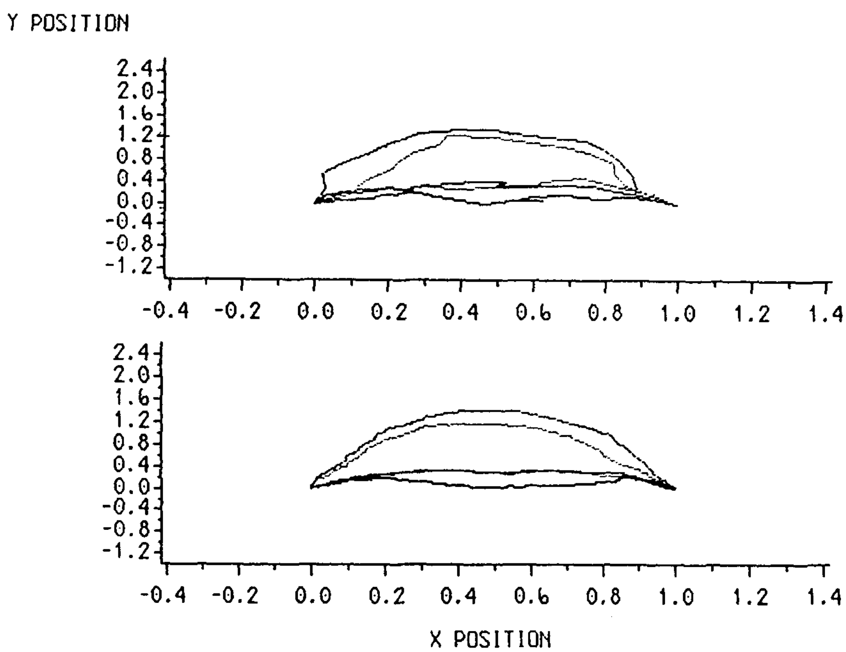

Figure 6: The solutions of suspension bridge cable and galloping cable model (from top to bottom) with $\mu = 4.8$ and $\lambda = 0.6$.

This was unexpected, because the cable is subject to vertical small forcing. The

15-particle cases have more pronounced longitudinal motions than the 63-particle cases. Figure 7 shows longitudinal motions when $\mu = 4.6$ and $\lambda = 2.0$.

4. Some Concluding Remarks

In the third section, we mainly discussed about the common and different results of large amplitude solutions between the more discrete case with fewer particles and the more continuous case with more particles. These result from careful and exhaustive computational experiments[5]. Of course, only some of them are covered here.

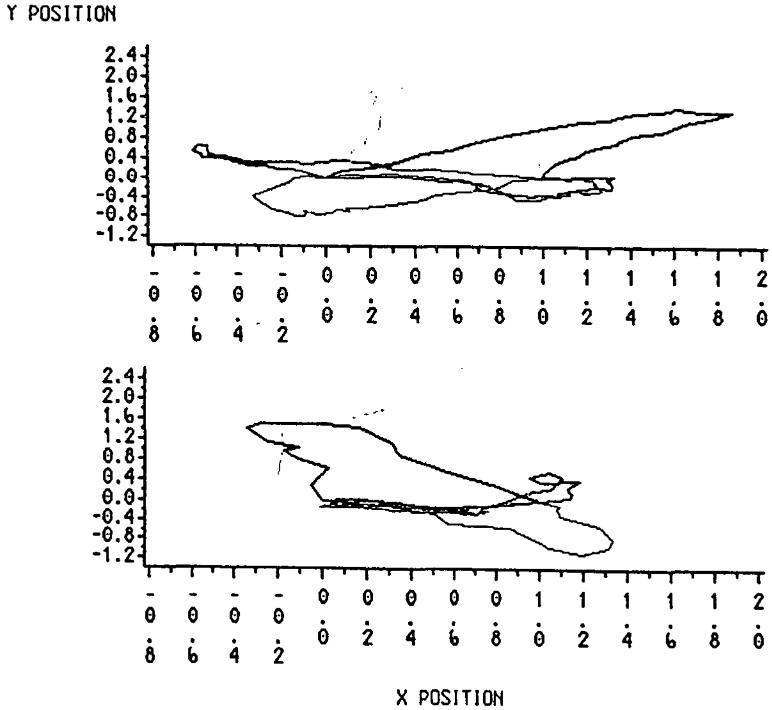

Figure 7: The longitudinal motions of suspension bridge cable and galloping cable model (from top to bottom) with $\mu = 4.6$ and $\lambda = 2.0$.

The phenomenon commonly found regardless of particles is that the long term behavior of the system is highly dependent on the initial conditions over a wide range of frequencies and amplitudes. If the initial conditions are close to equilibrium, the solution will be close to a linear small amplitude solution. On the other hand, if there is a large initial displacement, the solution after large time may either converge to the small linear solution or may remain in a large amplitude solution.

We are unsure at this point whether there are large-amplitude solutions in the galloping cable which are symmetric and of the same period, even though such solution was found in the suspension bridge cable, as shown in figure 3. This may become unstable in certain regions of the parameter space, which could explain the transition to the unsymmetric solutions. Further investigation of the bifurcation of this system will undoubtedly yield additional information.

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