

ON THE CONFORMAL DEFORMATION OVER WARPED PRODUCT MANIFOLDS

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ABSTRACT. Let $(M = B \times_f F, g)$ be an $(n \geq 3)$ -dimensional differential manifold with Riemannian metric g . We solve the following elliptic nonlinear partial differential equation

$$\frac{4(n-1)}{n-2} \Delta_g u(x) - h(x)u(x) + H(x)u(x)^{\frac{n+2}{n-2}} = 0,$$

where Δ_g is the Laplacian in the g -metric and $h(x)$ is the scalar curvature of g and $H(x)$ is a function on M .

1. Introduction

Let (M, g) be an $n(\geq 3)$ -dimensional differential manifold with Riemannian metric g . We say that another Riemannian metric \bar{g} on the given manifold is conformal to g if there exists a positive function $u(x)$ on M such that $\bar{g} = u^{\frac{4}{n-2}}g$. If $H(x)$ is a function on M , then we naturally ask : Can we find a function $u(x)$ on M such that \bar{g} is conformal to g and $H(x)$ is the scalar curvature of \bar{g} ? This question is equivalent to the problem of solving the elliptic nonlinear partial differential equation

$$\frac{4(n-1)}{n-2} \Delta_g u - h(x)u + H(x)u^{\frac{n+2}{n-2}} = 0, u > 0, \quad (1.1)$$

where Δ_g is the Laplacian in the g - metric and $h(x)$ is the scalar curvature of g (cf. [K.W.1,2,3], [A,p.126], or [N],etc.).

In particular, in this paper, we consider the conformal deformation of metrics on some warped product manifold $M = B \times_f F$ (cf. Definition 2.1). The concept

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of the warped product manifold is important not only in Riemannian geometry but also in Lorentzian geometry. In Riemannian geometry, the warped product is used for studying manifolds with various curvatures (cf.[B.O.], [D.G.], [D.D.], [Eb], [Ej], [K.K.P.], [M.M.]). And there is other application for the cohomology theory (cf. [Z]). And in Lorentzian geometry, for example, Minkowski spacetime, Schwarzschild spacetime and the Robertson-Walker spacetime are well-known examples of Lorentzian warped product manifolds(cf.[B.E.],[B.E.P.],[O.]).

Although throughout this paper we will assume that all data (M , metric g , and curvature, etc.) are smooth, this is merely for convenience. Our proofs go through with little or no change if one makes minimal smoothness hypotheses. For example, without changing any proofs we need only assume that the given data are Holder continuous.

2. Preliminaries on a warped product manifold

In this section, we briefly recall some results on a warped product manifold. Complete details may be found in [B.E.], [B.O.], or [O].

On a Riemannian product manifold $B \times F$, let π and σ be the first and second projections of $B \times F$ onto B and F , respectively, and let $f > 0$ be a smooth function on B .

Definition 2.1. *The warped product manifold $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where g_B and g_F are metric tensors of B and F , respectively. In other words, if v is tangent to M at (p, q) , then

$$g(v, v) = g_B(d\pi(v), d\pi(v)) + f^2(p)g_F(d\sigma(v), d\sigma(v)).$$

Here B is called the base of M and F the fiber. And we denote the metric g by \langle, \rangle . In view of the following (1) in [Remark 2.2] and Lemma 2.3, we also denote the metric g_B by \langle, \rangle and the metric g_F by $(,)$.

[Remark 2.2]. Now we list some elementary properties of the warped product manifold $M = B \times_f F$. (For details, see [B.E.], [B.O.] or [O]).

- (1) For each $q \in F$, the map $\pi|_{\sigma^{-1}(q)=B \times q}$ is an isometry onto B .
- (2) For each $p \in B$, the map $\sigma|_{\pi^{-1}(p)=p \times F}$ is a positive homothetic map onto F with homothetic factor $1/f(p)$.
- (3) For each $(p, q) \in M$, the horizontal leaf $B \times q$ and the vertical fiber $p \times F$ are orthogonal at (p, q) .
- (4) The horizontal leaf $\sigma^{-1}(q) = B \times q$ is a totally geodesic submanifold of M and the vertical fiber $\pi^{-1}(p) = p \times F$ is a totally umbilic submanifold of M .
- (5) If ϕ is an isometry of F , then $1 \times \phi$ is an isometry of M . And if ψ is an isometry of B such that $f = f \circ \psi$, then $\psi \times 1$ is an isometry of M .

Lemma 2.3. *If h is a smooth function on B , then the gradient of the lift $h \circ \pi$ of h to M is the lift to M of gradient of h on B .*

Proof. See Lemma 7.34 in [O].

If there is no confusion, we simplify the notations by writing h for $h \circ \pi$ and $grad(h)$ for $grad(h \circ \pi)$. And for a covariant tensor A on B , its lift \bar{A} to M is just its pullback $\pi^*(A)$ under the projection $\pi : M \rightarrow B$. That is, if A is $(1,s)$ -tensor, and if $v_1, \dots, v_s \in T_{(p,q)}M$, then $\bar{A}(v_1, \dots, v_s) = A(d\pi(v_1), \dots, d\pi(v_s)) \in T_p(B)$. Hence if v_k is vertical, then $\bar{A} = 0$ on B . For example, if f is a smooth function on B , the lift to M of the Hessian of f is also denoted by H^f . This agrees with the Hessian of the lift $f \circ \pi$ generally only on horizontal vectors. For detail computation, see Lemma 5.1 in [B.E.P.].

On the given warped product manifold $M = B \times_f F$, we also write S^B for the pullback by π of the scalar curvature S_B of B and similarly for S^F . From now on, we denote $grad(f)$ by ∇f .

Theorem 2.4. *If S is the scalar curvature of $M = B \times_f F$ with $n = dim F > 1$, then*

$$S = S^B + \frac{S^F}{f^2} - 2n \frac{\Delta f}{f} - n(n-1) \frac{\langle \nabla f, \nabla f \rangle}{f^2}, \quad (2.1)$$

where Δ is the Laplacian on B .

Proof. For detail computation, see Theorem 2.5 in [E.J.K.] or [O.p.214].

Here we can ask the following question: if $S_F(q) \equiv c$ (constant) on F , can we find the warping function $f > 0$ on B such that the warped metric g admits $S(p, q)$ as the constant scalar curvature on $M = B \times_f F$? In case that $S(p, q) \equiv k$ for all

$(p, q) \in M$, then the equation (2.1) is the pullback by π of the following equation, i.e.,

$$k = S_B(p) + \frac{c}{f^2} - 2n \frac{\Delta f}{f} - n(n-1) \frac{\langle \nabla f, \nabla f \rangle}{f^2},$$

that is,

$$\Delta f + \frac{1}{2n}(k - S_B)f - \frac{c}{2nf} + \frac{n-1}{2} \frac{\langle \nabla f, \nabla f \rangle}{f} = 0. \quad (2.2)$$

3. Main result

In this section, we restrict our results to the case that $B = (a, b)$ is an open connected subset of R with the positive definite metric dt^2 and $-\infty \leq a < b \leq +\infty$. Recalling that $\Delta f = f''(t)$ and $\langle \nabla f, \nabla f \rangle = (f'(t))^2$, and $S_B \equiv 0$ and by the change of variable $f(t) = \sqrt{v(t)}$, we have the following equation from the equation (2.2),

$$v''(t) + \frac{(n-3)}{4} \frac{|v'(t)|^2}{v(t)} + \frac{k}{n}v(t) - \frac{c}{n} = 0, \quad (3.1)$$

where we assume that F is a Riemannian manifold with a constant scalar curvature c and $\dim F = n > 1$. (Also, see another computation in [B.E., p. 78] and note that, in Lorentzian case, there is the difference of sign).

Theorem 3.2. *If $\dim F = n = 3$, and F is a compact 3-dimensional Riemannian manifold, then for any real number k we have the positive warping function $v(t)$:*

$$\begin{aligned} i) k > 0, v(t) &= c_1 \sin(\sqrt{\frac{k}{n}}t) + c_2 \cos(\sqrt{\frac{k}{n}}t) + \frac{c}{k}, \\ ii) k = 0, v(t) &= \frac{c}{2n}t^2 + c_1t + c_2, \\ iii) k < 0, v(t) &= c_1 \sinh(\sqrt{\frac{-k}{n}}t) + c_2 \cosh(\sqrt{\frac{-k}{n}}t) + \frac{c}{k}, \end{aligned}$$

where c_1 and c_2 are suitable constants.

Proof. If $n = 3$, then we have a simple differential equation,

$$v''(t) + \frac{k}{n}v(t) - \frac{c}{n} = 0.$$

Putting $h(t) = \frac{k}{n}v(t) - \frac{c}{n}$, it follows that $h''(t) + \frac{k}{n}h(t) = 0$. Hence we can see easily that we have the above solutions from the elementary method of the ordinary differential equation.

Here we carefully choose the above coefficients so that $v(t)$ is positive on (a, b) . For example, if $B = R$, the case ii) should satisfy that either $c > 0$ and $c_1^2 - 2\frac{c}{n}c_2 < 0$ or $c = c_1 = 0$ and $c_2 > 0$. Even though there are many considerations about the solutions according to cases i), ii), and iii), we omit here.

From now on, we assume that for the warped product $M = B \times_v F$, $B = (a, b)$ with $-\infty \leq a < b \leq +\infty$ and F is a compact 3-dimensional Riemannian manifold with a constant scalar curvature. Then, according to Theorem 3.1, we can always find a warping function $v(t)$ so that, for any real number k , the warped product manifold $M = B \times_v F$ admits a constant scalar curvature k .

Naturally we ask the following question:

Question (A): Assume that $(M = B \times_v F, g)$ satisfies the above assumptions. And let $H(t, x)$ be a smooth function on M . Then does there exist a new metric \bar{g} on M such that \bar{g} is conformal to g (i.e., $\bar{g} = u(t, x)^2g$) and $H(t, x)$ is a scalar curvature of \bar{g} ?

Recalling that $\dim M = 4$, according to the equation (1.1), Question (A) is equivalent to the problem of solving the elliptic nonlinear partial differential equation

$$6\Delta_g u(t, x) - ku(t, x) + H(t, x)u(t, x)^3 = 0, u(t, x) > 0, \quad (3.2)$$

where Δ_g is a Laplacian in the g -metric on $M = B \times_v F$ and k is the constant scalar curvature of g .

Theorem 3.2. *Let $\Phi : M = B \times_v F \rightarrow R$ be a smooth function of the form $\Phi = \phi_1(t)\phi_2(x)$, where $\phi_1 : (a, b) \rightarrow R$ and $\phi_2 : F \rightarrow R$ are smooth. Then*

$$\Delta_g \Phi = [\phi_1''(t) + \frac{3}{2v(t)}\phi_1'(t)v'(t)]\phi_2(x) + \frac{\phi_1(t)}{v(t)}\Delta_{g_F}\phi_2(x),$$

where Δ_{g_F} is a Laplacian on F in the g_F -metric.

Proof. See Theorem 5.4 in [B.E.P.].

Theorem 3.3. *If $\int_F K(x)dV < 0$, then there exists $r_0 > 0$ such that one can solve the equation $6\Delta_{g_F}\phi_2(x) + r\phi_2(x) + K(x)\phi_2(x)^3 = 0$, $\phi_2(x) > 0$, for $0 < r < r_0$, but not for $r > r_0$.*

Proof. See Proposition 4.8 in [K.W.1]

If $\phi_1(t)$ is the solution of a linear ordinary differential equation

$$\phi_1''(t) + \frac{3v'(t)}{2v(t)}\phi_1'(t) - \frac{k}{6}\phi_1(t) - r\frac{\phi_1(t)}{6v(t)} = 0,$$

for small $r > 0$, then we can find $\phi_1(t)$ which is positive. This is possible because, by the elementary ordinary differential equation theorem, we can choose the suitable initial conditions about $\phi_1(x_0)$ and $\phi_1'(x_0)$ for some point $x_0 \in (a, b)$ so that $\phi_1(x)$ is positive in (a, b) , maybe, in a smaller domain. If so, at first, we choose the domain as ours. Therefore, we have the following theorem.

Theorem 3.4. *Let $\phi_1(t)$ be the positive solution of a linear ordinary differential equation $\phi_1''(t) + \frac{3v'(t)}{2v(t)}\phi_1'(t) - \frac{k}{6}\phi_1(t) - r\frac{\phi_1(t)}{6v(t)} = 0$ for small $r > 0$ and let $H(t, x)$ be a function of the form $\frac{K(x)}{\phi_1(t)^2v(t)}$, where $\int_F K(x)dV < 0$. Then there exists a new metric \bar{g} on $M = (a, b) \times_v F$ such that \bar{g} is conformal to g and $H(t, x)$ is a scalar curvature of \bar{g} .*

Proof. We have only to show that there exists a solution of the equation (3.2). Put $u(t, x) = \phi_1(t)\phi_2(x)$. Theorem 3.2 and Theorem 3.3 imply the following:

$$\begin{aligned} & 6\Delta_g u(t, x) - ku(t, x) + H(t, x)u(t, x)^3 \\ &= 6\Delta_g(\phi_1(t)\phi_2(x)) - k(\phi_1(t)\phi_2(x)) + H(t, x)(\phi_1(t)\phi_2(x))^3 \\ &= 6\left[\phi_1''(t) + \frac{3}{2v(t)}\phi_1'(t)v'(t)\right]\phi_2(x) + 6\frac{\phi_1(t)}{v(t)}\Delta_{g_F}\phi_2(x) - k(\phi_1(t)\phi_2(x)) \\ &+ H(t, x)(\phi_1(t)\phi_2(x))^3 \\ &= 6\left[\phi_1''(t) + \frac{3}{2v(t)}\phi_1'(t)v'(t) - \frac{k}{6}\phi_1(t) - r\frac{\phi_1(t)}{6v(t)}\right]\phi_2(x) \\ &+ \frac{\phi_1(t)}{v(t)}[6\Delta_{g_F}\phi_2(x) + r\phi_2(x) + K(x)\phi_2(x)^3] \\ &= 0. \end{aligned}$$

[*Remark 3.5*]. Little is known about the more general form of $H(t, x)$. And if $\int_F K(x)dV \geq 0$, we have some partial results. But in this case there is some obstructions (cf.[J.2])

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