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THE LAW OF COSINES IN A TETRAHEDRON

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ABSTRACT. We will construct the generalized law of cosines in a tetrahedron, in a natural way, which gives three dimensional Pythagoras' theorem and enables us to calculate the volume of an arbitrary tetrahedron.

1. Introduction

The law of cosines in a triangle says: The square of any side is equal to the sum of the squares of the other two sides minus twice the product of those sides times the cosine of the angle between them. It is interesting to construct a parallelogram with this law to the case of a tetrahedron. The purpose of this paper is to study the law of cosines in a tetrahedron which gives several kinds of formulas for volume and area of it as applications.

The law of cosines in a tetrahedron in words is: The square of any triangle is equal to the sum of the squares of the other three triangles minus the sum of twice the product of other two triangles of those triangles times the cosine of the angle between them.

In this paper, we get the law of cosines in two similar triangles and tetrahedrons which give generalized Pythagoras' theorem in a triangle and a tetrahedron. We also examine the volume of a tetrahedron which is a parallelogram with approximately the same as the area of a triangle.

2. The law of cosines

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We consider a triangle $A_1A_2A_3$ with sides A_2A_3, A_3A_1, A_1A_2 and corresponding lengths l_1, l_2, l_3, respectively. Let θ_i for $i = 1, 2, 3$ be the angle corresponding to vertex A_i. Then we know

$$l_1 = l_2 \cos\theta_3 + l_3 \cos\theta_2 \tag{A.1}$$
$$l_2 = l_1 \cos\theta_3 + l_3 \cos\theta_1 \tag{A.2}$$
$$l_3 = l_1 \cos\theta_2 + l_2 \cos\theta_1 \tag{A.3}$$

and the law of cosines in a triangle

$$l_1^2 = l_2^2 + l_3^2 - 2l_2l_3 \cos\theta_1 \tag{B.1}$$
$$l_2^2 = l_1^2 + l_3^2 - 2l_1l_3 \cos\theta_2 \tag{B.2}$$
$$l_3^2 = l_1^2 + l_2^2 - 2l_1l_2 \cos\theta_3 \tag{B.3}$$

Now we consider two similar triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ with corresponding lengths l_1, l_2, l_3 and l_1', l_2', l_3', respectively. And let θ_i and θ_i' for $i = 1, 2, 3$ be the corresponding angles. The following proposition gives a generalized form of Pythagoras' theorem in two similar triangles with θ_1 right angle which says:

$$l_1l_1' = l_2l_2' + l_3l_3'$$

Proposition 2.1.

$$l_1l_1' = l_2l_2' + l_3l_3' - (l_2l_3' + l_2'l_3)\cos\theta_1.$$

Proof. The proof is a straightforward application of the law of cosines in a triangle by ratio of similarity times. □

We now will turn our attention to an arbitrary tetrahedron $A_1A_2A_3A_4$ with triangles $A_2A_3A_4, A_1A_3A_4, A_1A_2A_4, A_1A_2A_3$ labeled by T_1, T_2, T_3, T_4 and corresponding areas s_1, s_2, s_3, s_4, respectively. Dihedral angle corresponding to T_i and T_j for any $i, j = 1, 2, 3, 4$ is denoted by θ_{ij}.

Then we establish the following lemma which is the generalization of (A.1)-(A.3) to a tetrahedron.

Lemma 2.2. *For any $k = 1, 2, 3, 4$ we have*

$$s_k = \sum_{\substack{i \neq k \\ 1 \leq i \leq 4}} s_i \cos \theta_{ki}. \tag{$C.k$}$$

Proof. It is enough to prove when $k = 1$. Assume first that all θ_{1i}'s for $i = 2, 3, 4$ are acute. Drop a perpendicular A_1A' from vertex A_1 to opposite triangle $A_2A_3A_4$. Since the area s_1 is the sum of three triangles $A'A_3A_4, A'A_2A_4, A'A_2A_3$ which are projections of triangles $A_1A_3A_4, A_1A_2A_4, A_1A_2A_3$ with respect to corresponding dihedral angles $\theta_{12}, \theta_{13}, \theta_{14}$ respectively. So we have

$$s_1 = s_2 \cos \theta_{12} + s_3 \cos \theta_{13} + s_4 \cos \theta_{14}$$

Next, to prove it for the obtuse angle case, it is sufficient to assume that θ_{14} is obtuse. Drop a perpendicular A_1A'' from vertex A_1 to opposite plane containing triangle $A_2A_3A_4$. Then the area s_1 is given by the areas of triangles $A''A_3A_4, A''A_2A_4, A''A_2A_3$ as the follow:

$$\begin{aligned} s_1 &= s_2 \cos \theta_{12} + s_3 \cos \theta_{13} - s_4 \cos(\pi - \theta_{14}) \\ &= s_2 \cos \theta_{12} + s_3 \cos \theta_{13} + s_4 \cos \theta_{14} \end{aligned}$$

which completes the proof. □

In an arbitrary triangle, the length of one side is less than the sum of the lengths of the other two sides. The same version of arbitrary tetrahedron follows as well. Since any dihedral angle θ satisfies $0 < \theta < \pi$ and $|\cos \theta| < 1$, by Lemma 2.2 and triangle inequality, we obtain the following fact.

Remark. *In a tetrahedron the area of one triangle is less than the sum of the areas of the other three triangles.*

Moreover we obtain the law of cosines in a tetrahedron, which is the generalization of (B.1)-(B.3).

Theorem 2.3. *For any $k = 1, 2, 3, 4,$ we have*

$$s_k^2 = \sum_{\substack{j \neq k \\ 1 \leq j \leq 4}} s_j^2 - 2 \sum_{\substack{i,j \neq k \\ 1 \leq i,j \leq 4}} s_i s_j \cos \theta_{ij} \tag{$D.k$}$$

Proof. From Lemma 2.2 and the fact that $\theta_{ij} = \theta_{ji}, i, j = 1, 2, 3, 4$
$(C.1) \times s_1 - (C.2) \times s_2 - (C.3) \times s_3 - (C.4) \times s_4$ follows

$$s_1^2 = s_2^2 + s_3^2 + s_4^2 - 2s_2s_3\cos\theta_{23} - 2s_3s_4\cos\theta_{34} - 2s_4s_2\cos\theta_{42}.$$

Similarly, for $k = 2, 3, 4$, $(C.k) \times s_k - \sum_{\substack{j\neq k\\ 1\leq j\leq 4}} (C.j) \times s_j$ follows the above result. □

Here we can obtain the following result which is the generalized form of the law of cosines in two similar tetrahedrons which gives a generalized Pythagoras' theorem in two similar tetrahedrons with three right dihedral angles. Consider two simillar tetrahedrons with corresponding areas of its triangles, s_i and s_i' for $i = 1, 2, 3, 4$.

Corollary 2.4.

$$\begin{aligned} s_1s_1' =& s_2s_2' + s_3s_3' + s_4s_4' - (s_2s_3' + s_2's_3)\cos\theta_{23} \\ & - (s_3s_4' + s_3's_4)\cos\theta_{34} - (s_2s_4' + s_2's_4)\cos\theta_{24} \end{aligned}$$

Proof. It is an easy consequence of (D.k) by ratio of similarity times. □

3. Applications of the law of cosines

Here we will calculate the volume of an arbitrary tetrahedron and derive several equations for the area and volume of special tetrahedrons.

As we have already noted, consider a triangle $A_1A_2A_3$ with lengths l_1, l_2, l_3 and θ_i for $i = 1, 2, 3$, the angle corresponding to vertex A_i. We know that the area s is given by

$$s = \frac{1}{2}l_1l_2\sin\theta_3 = \frac{1}{2}l_2l_3\sin\theta_1 = \frac{1}{2}l_1l_3\sin\theta_2.$$

Similarly, for an arbitrary tetrahedron $A_1A_2A_3A_4$ the volume v is given by the following proposition.

Proposition 3.1. *For any* $i, j = 1, 2, 3, 4$

$$v = \frac{2}{3l_{ij}}s_is_j\sin\theta_{ij}$$

where l_{ij} *denotes the length of common side of* T_i *and* T_j.

Proof. Let A_1A_1' be a perpendicular from vertex A_1 to opposite plane $A_2A_3A_4$ extended if necessary and A_1A_1'' a perpendicular from vertex A_1 to side A_3A_4. Then by the theorem of three perpendiculars $A_1'A_1''$ is perpendicular to side A_3A_4. So, we have $h_1 = h_2 \sin\theta_{12}$, where h_1 and h_2 are lengths of A_1A_1' and A_1A_1'', respectively. As required,

$$v = \frac{1}{3}s_1h_2\sin\theta_{12} = \frac{2}{3l_{12}}s_1s_2\sin\theta_{12}$$

follows from that $v = \frac{1}{3}s_1h_1$ and $s_2 = \frac{1}{2}l_{12}h_2$. as required. The rest of the argument is identical with the above case. □

The proceding proposition says that the volume of a tetrahedron is given by an angle, a length and areas just as the area of a triangle.

Example 3.2. *In three dimension the region defined by*

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} < 1, \quad x > 0, y > 0, z > 0$$

is a tetrahedron $OABC$ with three of its faces being right- angled triangles T_1, T_2, T_3 in the coordinate planes. And its forth face T_4 has vertices at $A(a,0,0), B(0,b,0)$ and $C(0,0,c)$. Label the areas of triangle T_i, $i = 1,2,3,4$ as s_i. Then, from Theorem 2.3, we have a remarkable result which is known as three dimensional Pythagoras' theorem:

$$s_4^2 = s_1^2 + s_2^2 + s_3^2.$$

By Proposition 3.1 the volume v of a tetrahedron $OABC$ is given by the following,

$$v = \frac{2}{3a}\frac{ab}{2}\frac{ac}{2} = \frac{abc}{3!}$$

which is just the generalized form of the area of right-angled triangle.

We conclude this paper with a final example which establishes the relationships among angles, lengths, areas and volume of special tetrahedrons.

Example 3.4. *Consider a triangular pyramid T whose base plane T_1 is regular triangle with side length a and area s. Suppose that all hypotenuse triangles T_i, $i = 2,3,4$ are isosceles congruent ones with area t and length b of legs of them. Let α and β denote dihedral angles generated by base-side face and side-side face, respectively. Then, by the law of cosines, we have the followings:*

$$
\begin{aligned}
&s = 3t\cos\alpha \\
&t = s\cos\alpha + 2t\cos\beta \\
&s^2 = 3t^2(1 - 2\cos\beta) \\
&v = \frac{2}{3a} st\sin\alpha = \frac{2}{9a} s\sqrt{9t^2 - s^2} = \frac{1}{12} a^2\sqrt{3b^2 - a^2}
\end{aligned}
$$

Notice what happens when all T_i's are congruent so that T becomes a regular tetrahedron generated by four regular triangles with side length a and dihedral angle α. The above facts give us the well known facts:

$$
\cos\alpha = \frac{1}{3}, \quad v = \frac{\sqrt{2}}{12} a^3.
$$

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