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Engineering Economy Interpretation of Economic Production Cycles in an Imperfect Production System
- 불완전한 생산체계의 경제적 생산주기에 관한 경제성공학적 해석 -

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요 지

본 논문에서는 제품을 생산하는 도중에 생산체계의 상태가 관리상태에서 이상상태로 전이될 수 있는 불완전한 생산체계에 있어서의 경제적 생산주기 결정모형을 다룬다. 생산체계가 관리상태에 머무는 생산시간이 지수분포를 따른다는 가정하에서 전체 현금흐름의 현재가치를 생산주기의 함수로 유도하고, 이 함수를 최대화하는 경제적 생산주기의 근사해를 구한다. 근사해에서 출발하여 최적해를 찾아 내는 간단한 알고리즘을 개발하고, 이를 적용한 수치예를 보인다.

1. Introduction

In the inventory literature, the economic production quantity model has been analyzed under various conditions (See Hax and Candea[5]). The majority of these researches are based on the approach that minimizes the average cycle costs ignoring the time value of money. As discounted cash flow analysis is strongly recommended and often used in capital budgeting and investment analysis, the economic production quantity models can also be formulated and analyzed within the net present value framework.

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Problems of this type have been discussed by several authors, including Dohi et al.[4], Chung[1], Chung and Lin[2,3], and Moon and Yun[8,9]. All of these researches are based on the assumption that production process always produces items of acceptable quality. However, as Lee and Park[6] indicated, production quality is not always perfect, and is usually dependent on the state of the production process.

In this paper, we analyze the economic production quantity model in an imperfect production system based on the net present value framework. This may be essentially an extension of the models analyzed by Chung[1] and Lee and Rosenblatt[7]. We formulate the given problem, and propose a simple algorithm to compute the optimal production cycle time.

2. Problem formulation

Consider the production of a single item on a single machine or production process. The usual assumptions of the classical economic production quantity model are also used here. That is, we assume that demand is constant and continuous and that all the demand must be met. In addition to the classical assumptions, we assume that although the production process starts to produce a lot in an 'in-control' state, it shifts to an 'out-of-control' state after exponentially distributed production time of mean $1/\mu$. As Lee and Rosenblatt[7] described, the exponential assumption for the elapsed time of the 'in-control' state are commonly founded in the quality control literature.

For convenience, we use notation similar to Lee and Rosenblatt[7]. The following notation will be used as parameter.

P = the production rate in units per year,
D = the demand rate in units per year,
K = set-up cost for each production run including the restoration of the production process,
C = unit production cost,
I = unit selling price
S = cost incurred by producing a defective item, such as cost of rework, loss of goodwill, etc.,
α = percentage of defective units produced when the process is in the out-of-control state,
r = interest rate of a continuous type.

Decision variables are as follows :
Q = the production quantity,
T = the cycle time for each production lot.

A discounted cash flow approach is adopted to analyze and optimize the discounted cost in a given situation.

Present Value of Cash Flows for the First Cycle

As actual production time is DT/P, the present value of the production cost is obtained by

$$\int_0^{DT/P} CPe^{-rt}\,\mathrm{d}t = \frac{CP}{r}[1-e^{-r\frac{DT}{P}}] \tag{1}$$

The present value of the income is obtained by

$$-\int_0^{T} IDe^{-rt}\,\mathrm{d}t = -\frac{ID}{r}[1-e^{-rT}] \tag{2}$$

Define τ to be the elapsed time of a shift from the in-control state to the out-of-control state since the beginning of the production. Then, present value of the cost incurred by the defective items is

$$\begin{aligned} &\int_0^{\frac{DT}{P}}\int_\tau^{\frac{DT}{P}} SaPe^{-rt}\,\mathrm{d}t\,\mu e^{-\mu\tau}\,\mathrm{d}\tau \\ &= \frac{\mu SaP}{r}\left[e^{-(r+\mu)\frac{DT}{P}}\left(\frac{1}{\mu}-\frac{1}{r+\mu}\right)-e^{-r\frac{DT}{P}}\frac{1}{\mu}+\frac{1}{r+\mu}\right] \end{aligned} \tag{3}$$

Thus, from the equations (1), (2), and (3), the present value of cash flows for the first cycle is

$$\begin{aligned} FC(T) &= K+\frac{CP}{r}[1-e^{-r\frac{DT}{P}}]+\frac{ID}{r}[1-e^{-rT}] \\ &\quad \frac{\mu SaP}{r}\left[e^{-(r+\mu)\frac{DT}{P}}\left(\frac{1}{\mu}-\frac{1}{r+\mu}\right)-e^{-r\frac{DT}{P}}\frac{1}{\mu}+\frac{1}{r+\mu}\right] \\ &= K+\frac{CP}{r}+e^{-r\frac{DT}{P}}[-\frac{CP}{r}-\frac{SaP}{r}] \\ &\quad +e^{-(r+\mu)\frac{DT}{P}}\left[\frac{SaP}{r}-\frac{SaP\mu}{r(r+\mu)}\right]+\frac{SaP\mu}{r(r+\mu)}-\frac{ID}{r}[1-e^{-rT}] \end{aligned} \tag{4}$$

Present Value Cash Flows of an Infinite Planning Horizon

Suppose that the typical production cycle described above is repeated infinitely, then the present value of the total cash flows is

$$TC(T) = FC(T)*[1+e^{-rT}+e^{-2rT}+\cdots\cdot]$$
$$= \frac{FC(T)}{[1-e^{-rT}]}$$

The first derivative of $TC(T)$ with respect to T, say $TC'(T)$ is as follows

$$TC'(T) = -\frac{e^{-rT}}{(1-e^{-rT})^2} f(T)$$

where

$$f(T) = (1-e^{rT})\left[De^{-r\frac{DT}{P}}(C+Sa)-DSae^{-(r+\mu)\frac{DT}{P}}\right]$$
$$+P\left[\frac{rK}{P}+C-e^{-r\frac{DT}{P}}(C+Sa)+Sa\frac{r}{r+\mu}e^{-(r+\mu)\frac{DT}{P}}+Sa\frac{\mu}{r+\mu}\right]$$

Note that $TC'(T)$ and $f(T)$ have the opposite signs. By setting $f(T)$ equal to 0, we can obtain the optimal cycle length T^*.

Proposition 1

$f(T) = 0$ has at least one solution between T = 0 and T = ∞.

proof

(1) $f(0) = rK > 0$

(2) $$\lim_{T\to\infty} f(T) = \lim_{T\to\infty}\left[-D(C+Sa)e^{r(1-\frac{D}{P})T}+DSae^{r(1-\frac{D}{P})T-\mu\frac{D}{P}T}\right]+P\left(\frac{rK}{P}+C+\frac{Sa\mu}{r+\mu}\right)$$

$$= -\left[\lim_{T\to\infty} De^{r(1-\frac{D}{P})T}\right]\left[\lim_{T\to\infty}\left\{(C+Sa)+Sae^{-\mu\frac{D}{P}T}\right\}\right]+P\left(\frac{rK}{P}+C+\frac{Sa\mu}{r+\mu}\right).$$

$$\rightarrow -\infty$$

By (1) and (2), $f(0) > 0$ and $f(\infty) < 0$. Thus $f(T) = 0$ has at least one solution between T = 0 and T = ∞. ■

<u>Proposition 2</u>

$f(T)$ is monotone decreasing function for all T > 0.

<u>proof</u>

$$
\begin{aligned}
f'(T) = & -re^{rT}\left[De^{-r\frac{DT}{P}}(C+S\alpha)-DS\alpha e^{-(r+\mu)\frac{DT}{P}}\right] \\
& +(1-e^{rT})\left[-r\frac{D^2}{P}(C+S\alpha)e^{-r\frac{DT}{P}}+\frac{(r+\mu)S\alpha D^2}{P}e^{-(r+\mu)\frac{DT}{P}}\right] \\
& +rDe^{-r\frac{DT}{P}}(C+S\alpha)-S\alpha rDe^{-(r+\mu)\frac{DT}{P}}
\end{aligned}
$$

$$= (1-e^{rT})rDe^{-r\frac{DT}{P}}\cdot g(T)$$

$$\text{where} \quad g(T) = (C+S\alpha)(1-\frac{D}{P})-S\alpha\left[1-\frac{(r+\mu)D}{P}\right]e^{-\mu\frac{D}{P}T}$$

As $(1-e^{rT})$ is always negative for all T > 0, $f'(T)$ has the opposite sign to $g(T)$. By the way,

$$
\begin{aligned}
g(0) &= (C+S\alpha)(1-\frac{D}{P})-S\alpha\left[1-\frac{(r+\mu)D}{P}\right] \\
&= C(1-\frac{D}{P})+S\alpha\frac{D\mu}{\Pr} \quad > 0
\end{aligned}
$$

$$\lim_{T\to\infty} g(T) = (C+S\alpha)(1-\frac{D}{P}) \quad > 0$$

Moreover, as

$$g'(T) = S\alpha\mu\frac{D}{P}\left[1-\frac{(r+\mu)D}{P}\right]e^{-\mu\frac{D}{P}T},$$

$g(T)$ is monotone (decreasing or increasing) function. Thus, $g(T)$ is always positive and $f'(T)$ is always negative. This complete the proof. ■

By Proposition 1 and 2, $f(T)$ has a unique solution T^* > 0 and $TC(T)$ has the minimum value at this point.

3. Solution Algorithm and Numerical Example

Note that r is the interest rate of a continuous type and μ is failure rate when production is in progress. Thus, in most cases r and μ are very small compared with 1 so that it is reasonable to approximate the exponential functions in $f(T)$ as follows.

$$e^{-\mu x} \cong 1-\mu x+\frac{(\mu x)^2}{2!}$$

$$e^{-(r+\mu)x} \cong 1-(r+\mu)x+\frac{(r+\mu)^2 x^2}{2!}$$

Such an approximation has been frequently used in the quality control and engineering economy literature to simplify the mathematical analysis involved. For detail discussion of the above approximation, we can refer any Calculus textbook. For example, we can compute $e^{0.1}$ to an accuracy of 10^{-3} just with 3 terms in Taylor's formula. After some tedious mathematical manipulation using above McClaurin series approximation, $f(T)$ is approximated as follows.

$$\begin{aligned} f_{app}(T) &= \left[-\frac{Sa\mu rD^2}{2P}-\frac{Cr^2D}{2}(1-\frac{D}{P})\right]T^2+rK \\ &= -\frac{rD}{2}\left[\frac{Sa\mu D}{P}+Cr(1-\frac{D}{P})\right]T^2+rK \end{aligned}$$

Setting $f_{app}(T)$ equal to zero yields

$$T^*_{app} = \sqrt{\frac{2K}{D\left[\frac{Sa\mu D}{P}+Cr(1-\frac{D}{P})\right]}} \qquad (5)$$

On the ground of the Proposition 1 and 2, if we want to get the exact optimal solution, we can solve equation $f(T)$ = 0 numerically using bisection method as follows.

<u>Solution Algorithm</u>

Step 1 : Let $\varepsilon > 0$, $x_L = 0$, and $x_R = T^*_{app}$.
Step 2 : If $|f(x_R)| < \varepsilon$, $T^* = x_R$ and stop. Otherwise, go to step 3.
Step 3 : If $f(x_R) < 0$, go to step 4.
If $f(x_R) > 0$, set $x_L = x_R$ and $x_R = 2x_R$, and go to step 4.
Step 4 : $x^* = (x_L + x_R)/2$.
Step 5 : If $|f(x^*)| < \varepsilon$, $T^* = x^*$ and stop. Otherwise, go to step 6.
Step 6 : If $f(x^*) > 0$, set $x_L = x^*$. If $f(x^*) < 0$, set $x_R = x^*$. Go to step 4.

Numerical Example

Suppose that r = 0.2, P = 1000 (units per year), D = 600 (units per year), K = \$100, μ = 0.5, α = 0.7, S = \$0.5, C = \$5.
Then, by equation (5), we can obtain the approximated optimal production cycle time T^*_{app} = 0.812 directly.
Taking ε = 0.01 and following the above algorithm, we can obtain the exact optimal production cycle time T^* = 0.826 iteratively, which incurs total annual cost TC(0.826) = \$10472 (excluding income by letting I=0).
The cycle time becomes T = 0.913 by the well known classical model of perfect production system, which incurs total annual cost TC(0.913) = \$11172. We note that, if the optimal production cycle time of this paper is used instead of the classical model in our example, the cost saving is 6.27%.

4. Concluding Remark

In this paper, the economic production quantity model in an imperfect production system was analyzed based on the net present value framework. We formulated the given problem, derived an approximated solution, and proposed a simple algorithm to compute the optimal production cycle. This model improved the practicality of the assumption about the production system, which has been usually assumed to produce items of acceptable quality.

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