THE REMARK on THE SELF-SIMILAR SETS
-자기 동형 집합에 관하여-

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요 지

먼저 Cantor dust 의 성질 및 유사성, 축소인자, 불변집합, δ - covering, Box counting 차원 등에 대한 정의를 하였다, $\{f_i\}_{i=1}^{\infty}$ 를 $\mathbb{R}^n$상에서 개집합 조건을 만족시키는 축소인자 C_i에 대한 유사성 이라하자. F를 $\{f_i\}_{i=1}^{\infty}$ 에 대한 $\mathbb{R}^n$상의 불변집합, 즉, $F = \bigcup_{i=1}^{\infty} f_i(F)$를 만족시키는 집합이라 하자. 이때, $\sum_{i=1}^{\infty} C_i^s = 1$, $0 < C_i < 1$ 일 때, $\dim_H F = \dim_B F = s$ 임을 보임으로서, 자기동형집합의 후랙탈 차원에 대하여 논의 하고자 한다.

1. Introduction and preliminaries

Mandelbrot observed the study of the fractal from the existence of a ''Geometry of Nature ''. His studies have led us to think in a new scientific way about the edge of clouds. Fractal is to study the method of representation of many natural phenomena and provide the general framework for non-smooth and irregulars. The purpose of this paper

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is to prove some properties of fractals and study its dimension. The triadic Cantor set is the limit C of the sequence C_k of sets. We will define the limit to be the intersections C = $\bigcap_{k\in N} C_k$ is an official fractal. These are several properties of Cantor set ([1], [2], [10], [12]).

[1] Cantor set contains no intervals.

[2] Cantor set has no isolated points.

[3] Cantor set is closed ; that is, if a $\in \mathbb{R}$ has the property that every interval of the form ($a-\varepsilon$ $a+\varepsilon$) intersects C, then a $\in$ C.

We think functions as the following. Let (X,d) be a metric space. A self-map f : X→ X is called contraction mapping on X if there is a constant 0< c <1 such that d(f(x), f(y)) ≤ cd(x,y) for all x, y ∈ X. Such number c is called a contractivity factor for f.

(1-1) If d(f(x), f(y)) = cd(x,y), then f is called a similarity. Now let $\{ f_i\}_{i=1}^{\infty}$ be contractions on X. We call a subset A of X invariant for $\{ f_i\}_{i=1}^{\infty}$ if A = $\bigcup_{i=1}^{n} f_i(A)$. Such invariant sets are often fractals ([1] ,[11]). Let $A\subset \mathbb{R}^n$ and $\delta > 0$. A covering $\mu = \{ U_\alpha\}_{\alpha\in\Lambda}$ of A is called a δ-cover if each U_α is a set of diameter r_α ($0 < r_\alpha < \delta$). For s>0, define $H_\delta^s(A) = \inf \{ \sum_{\alpha\in\Lambda} | U_\alpha|^s ;\ \mu =\{ U_\alpha\}_{\alpha\in\Lambda}$ is a δ -covering of A }. Clearly, $H_\delta^s \le H_{\delta'}^k$, if $0 < \delta' < \delta$. Therefore, $\lim_{\delta\to 0} H_\delta^k(A)$ exist in the extended real number system.

1^0. Let μ be a mass distribution on $\mathbb{R}^n$ and let $F\subset \mathbb{R}^n$ be a bounded subset. For some $s \ge 0$, assume that there are numbers $c > 0$ and $\delta > 0$ such that $\mu(U) \le c\,|U|^s$ for each set U with and $|U| \le \delta$, then $H^s(F) \ge \frac{\mu(F)}{c}$ and $s \le \dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F$.

2^0. Let $\{ V_\alpha\} \in\Lambda$ be a collection of disjoint open subsets of $\mathbb{R}^n$ for which each V_α contains a ball of radius ar and is containd in a ball of radius br. Then any ball B of radius r intersects at most $(1+2b)^n\ a^n$ of the closures $\overline{V_i}$.

3^o. If $J_\sigma = [a, b]$, for $\sigma \in \{0, 1\}$, then set $J_{\sigma*0} = [a,\ a + x\,(b-a)]$ and $J_{\sigma*1} = [a + y(b-a),\ b]$, where the point x, y is chosen from the triangular region $\triangle = \{ (s,t)|\ 0\le s\le t\le 1 \}$ according to the uniform distribution.

4^0. Let $\mathbb{R}^m$ is a Euclidean space and a nonempty compact subset J is the closure of its interior in $\mathbb{R}^m$ and (Ω, Σ, P) is a probability space J = $\{ J_\sigma \mid \sigma \in N' = \bigcup_{n=0}^{\infty} N^n\}$ satisfying three properties. (1) $J_\phi(\omega)$ = J for almost all $\omega \in \Omega$, $\forall \sigma \in N'$, for allmost ω, if $J_\sigma(\omega)$ is nonempty, then $J_\sigma(\omega)$ is geometrically similar to J. (2) w: a point, for all $\sigma \in N'$, $J_{\sigma*1}(w)$, $J_{\sigma*2}(w)$, $J_{\sigma*3}(w)$, $\cdots$ is a sequence of nonoverlapping subsets of $J_\sigma(\omega)$. (3) The random vectors $\mathfrak{T}_\sigma$= $< T_{\sigma*1}, T_{\sigma*2}, T_{\sigma*3}, \cdots >$, $\sigma \in N^*$, are i.i.d., where $T_{\sigma*n}(\omega)$ equals the ratio of the diameter of $T_{\sigma*n}(\omega)$ to the diameter of $J_\sigma(w)$ if $J_\sigma(w)$ is nonempty.

2. Examples of Fractal Dimensions

First, we will show the dimension the triadic Cantor set. This example is known as the Cantor set. Mandelbrot has called it the Cantor set. Let C_0 be the closed unit interval [0, 1]. Then the set C_1 is obtained by removing the middle second from [0, 1] leaving $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$. The next set C_2 is defined by removing the middle second of the two intervals of C_1. This leaves $C_2 = [o, \frac{1}{16}] \cup [\frac{3}{16}, \frac{4}{16}] \cup [\frac{12}{16}, \frac{13}{16}] \cup [\frac{15}{16}, 1]$ and so on. Here, dimension of sets C_n using definition of similarity , is near to real number 0.5 and the dimension of the Sierpinski gasket is similarly. The example of the above become to fractals.

3. The Self-Similar Sets

Proposition 3.1. Let c_i be constants satisfying $0 < c_i < 1$ (i = 1, 2, 3, $\cdots$, n), and inf c_i is not zero. Then there is a unique nonnegative number s such that $\sum_{i=1}^{n} c_i^s = 1$ further, the number s is 0 if and only if $1 \leq i < \infty$.

Proof. [22] pp 248 - 253.

Theorem 3.2. Let $\{ f_i\}_{i=1}$ be the similarities on $\mathbb{R}^n$ with contractivity factors c_i which satisfy the open set condition on $\mathbb{R}^n$. If F is an invariant subset of $\mathbb{R}^n$

with respect to $\{ f_i \}_{i=1}^{\infty}$. i.e., $F = \bigcup_{i=1}^{\infty} f_i(F)$. Then $\dim_H F = \dim_B F = s$, where

(3- 1) $$\sum_{i=1}^{\infty} c_i^s = 1, \quad 0 < c_i < 1.$$

Proof. Assume that (3-1) holds. Put $A(i_1, i_2, \cdots, i_k) = (f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k})(A)$ for any subset $A \subset \mathbb{R}^n$. Let J_k be the set of all k-term sequences $(i_1, i_2, \cdots, i_k)$ with $1 \le i_j < \infty$. Then $F = \bigcup_{J_k} F(i_1, i_2, \cdots, i_k)$. Therefore

$$\sum_{J_k} | F(i_1, i_2, \cdots, i_k) |^s = \sum_{J_k} (c_{i_1} \cdots c_{i_k})^s |F|^s$$
$$= (\sum_{i_1} c_{i_1}^s) \cdots ((\sum_{i_k} c_{i_k}^s) |F|^s = |F|^s.$$

Choose k such that

$$| F(i_1, i_2, \cdots, i_k) | \le (\sup c_i)^k \le \delta$$

for any $\delta > 0$, then $H_\delta^s(F) \le |F|^s$ and so $H^s(F) \le |F|^s$. Let us consider the lower bound. Let I be the set of all infinite sequences $I = \{ (i_1, i_2, \cdots) : 1 \le i_j < \infty \}$, and let $I_{i_1, \cdots, i_k} = \{(i_1, \cdots, i_k, q_{k+1}, \cdots) : 1 \le q_j < \infty\}$ be the cylinder consisting of those sequences in I with initial terms $(i_1, i_2, \cdots, i_k)$. We define a mass distribution μ on I by $\mu(I_{i_1, \cdots, i_k}) = (c_{i_1} \cdots c_{i_k})^s$. Then since

$$(c_{i_1} \cdots c_{i_k})^s = \sum_{i=1}^{\infty} (c_{i_1} \cdots c_{i_k} c_i)^s,$$

$$\mu(I_{i_1, \cdots, i_k}) = \sum_{i=1}^{\infty} \mu(I_{i_1, i_2 \cdots i_k, i}).$$

Accordingly, μ is a mass distribution on subsets of I with $\mu(I) = 1$. Let us transfer μ to $\bar{\mu}$ on F by putting $\bar{\mu}(A) = \mu\{ (i_1, i_2, \cdots) ; x_{i_1, i_2, \cdots} \in A \}$ for each subset A of F, where $x_{i_1, i_2, \cdots} = \bigcap_{k=1}^{\infty} F(i_1, i_2, \cdots, i_k)$. Thus $\bar{\mu}(F) = 1$. The mass distribution $\bar{\mu}$ satisfies 1^0. By assumption, let $V \subset \mathbb{R}^n$ be a nonempty bounded open subset which satisfies the open set condition for $\{ f_i \}_{i=1}^{\infty}$. Since

$$\bar{V} \supset \bigcup_{i=1}^{\infty} f_i(\bar{V}), \quad f^k(\bar{V})$$

converges to F. Here f^k denote the k-times composition of f and the map f is defined by $f(A) = \bigcup_{i=1}^{\infty} f_i(A)$. In particular, $\bar{V} \supset F$ and

$$\bar{V}(i_1, i_2, \cdots, i_k) \supset F(i_1, i_2, \cdots, i_k)$$

for each finite sequence $(i_1, i_2, \cdots, i_k)$. Let $B = \{ B_r$ is any open ball | r is radious of $0 < r < 1\}$. We shall estimate $\bar{\mu}(B)$ by considering the sets $V(i_1, i_2, \cdots, i_k)$ with diameter comparable with that of B and with closure intersecting $F \cap B$. We are define the random set

$$K(\omega) = \bigcap_{i=1}^{\infty} \bigcup_{\sigma \in N^i} J_\sigma(\omega) = \bigcap_{i=1}^{\infty} \bigcup_{\sigma \in N^i} J_\sigma(\omega).$$

If K is nonempty with positive probability, K has Hausdorff dimension a, where a is the least $\beta > 0$. i.e. $E(\sum_{i=1}^{\infty} T_i^{\beta}) \le 1$. Let λ be m-dimensional Lebesgue measure and define a function $\phi : [0, \infty] \Rightarrow [0, \infty]$ by

$$\phi(\beta) = E(\sum_{i=1}^{\infty} T_i^{\beta}) = E(\sum_{i=1}^{\infty} T_{\sigma * i}^{\beta}), \text{ where } \sigma \in N^{'}.$$

The sets J_σ satisfies 4^0.

$$\sum_{n=1}^{\infty} \wedge(\inf(c_n)) \le \sum_{i=1}^{\infty} \lambda(\inf(J_i))$$

$$\le \lambda(\inf(J_i)) \le J_{i_1} J_{i_2} \cdots J_{i_k} \cdots \le cr, \text{ for all } c > 1.$$

Let Q denote the finite set of all sequences obtained in this way. Then there is exactly one value of k with $(i_1, i_2, \cdots, i_k) \in Q$. Since V(1), $\cdots\cdot$, V(m) are disjoint, $\{V(i_1, i_2, \cdots, i_k, l) ; 1 \le l \le m\}$ are disjoint for each $(i_1, i_2, \cdots, i_k) \in Q$ and so $\{V(i_1, i_2, \cdots, i_k, l) ; (i_1, i_2, \cdots, i_k) \in Q \text{ and } 1 \le l \le m\}$. Similarly, $F \subset \bigcup_Q F(i_1, i_2, \cdots, i_k) \subset \bigcup_Q \overline{V}(i_1, i_2, \cdots, i_k)$. Choose d_1 and d_2 so that V contains a ball of radius d_1 and is contained in a ball of radius d_2. Then, for each $(i_1, i_2, \cdots, i_k) \in Q$, the set $V(i_1, i_2, \cdots, i_k)$ contains a ball of radius $c_{i_1} c_{i_2} \cdots c_{i_k} d_1$ and therefore a ball of radius $(\inf_i c_i) d_1$r and is contained in a ball of radius $c_{i_1} c_{i_2} \cdots c_{i_k} d_2$ and hence in a ball of radius d_2 r. By Q_1 denote the set of those sequences $(i_1, i_2, \cdots, i_k)$ in Q such that B intersects $\overline{V}(i_1, i_2, \cdots, i_k)$. There are at most $q = (1 + 2d_2)^n d_1^{-n} (\inf_i c_i)^{-n}$ sequences in Q_1. Then

$$\overline{\mu}(B) = \overline{\mu}(F \cap B) \le \mu\{(i_1, i_2, \cdots\cdot) ; x_{i_1, i_2, \cdots\cdot} \subset F \cap B\}$$

$$\le \mu\{\bigcup_{Q_1} I_{i_1, i_2, \cdots, i_k}\}.$$

Since, if $x_{i_1, i_2, \cdots\cdot} \in F \cap B \subset \bigcup_{Q_1} \overline{V}(i_1, i_2, \cdots, i_k)$ then there is an integer k such that $(i_1, i_2, \cdots, i_k) \in Q_1$. Thus

$$\overline{\mu}(B) \le \sum_{Q_1} \mu(I_{i_1, i_2, \cdots, i_k}) = \sum_{Q_1} (c_{i_1} \cdots c_{i_k})^s \le \sum_{Q_1} r^s \le r^s q.$$

Since any set U is contained in a ball of radius $|U|$, $\overline{\mu}(U) \le |U|^s q$. Therefore, $H^s(F) \ge q^{-1} > 0$ by 2^0, and $\dim_H F = s$. Inductively, $\sum_Q (c_{i_1} \cdots c_{i_k})^s = 1$ by (3-1). If Q satisfying the condition of the Hausdorff metric, then Q contains at most $(\min_i c_i)^{-s} r^{-s}$ sequences. For each sequence $(i_1, i_2, \cdots, i_k) \in Q$,

$$|\overline{V}(i_1, i_2, \cdots, i_k)| = c_{i_1} \cdots c_{i_k} |\overline{V}| \le r |\overline{V}|$$

and so A is covered by $(\inf_i c_i)^{-s} r^{-s}$ sets of diameter $r|\overline{V}|$ for each r < 1. By largest number of disjoint balls of radius δ with center in F, $\overline{\dim}_B F \le s$, where s is the Hausdorff dimension.

4. Non-Integral Dimension of Irregular Sets

[$*_1$] Fractal Dimension

Definition 4.1. Let (X, d) be a complete metric space and let A $\subset$ X be a nonempty compact subset. For each $\varepsilon > 0$, let N(A, ε) denote the smallest number of closed balls of radius $\varepsilon > 0$ needed to cover A. If $D_A = \lim_{\varepsilon \to 0} \frac{\ln(N(A,\varepsilon))}{\ln(\frac{1}{\varepsilon})}$ exists, then D_A is called the box counting dimension of A. And we will also say "A has fractal dimension D_A". The intuitive idea behind fractal dimension is that a set A has fractal dimension D_A if N(A, ε) = $c \cdot \varepsilon^{-D_A}$ for some positive constant c. From this we obtain the following ([2], [12]) ;

$$D_A = \lim_{\varepsilon \to 0} \frac{\ln(N(A,\varepsilon)) - \ln c}{\ln(\frac{1}{\varepsilon})}.$$

[$*_2$] Hausdorff Dimension

Definition 3.3. $\dim_H F = \inf \{ s ;\ H^s(F) = 0 \} = \sup \{ s :\ H^s(F) = \infty \}$

$$H^s(F) = \begin{cases} \infty & \text{if } s \le \dim_H F \\ 0 & \text{if } s > \dim_H F \end{cases}.$$

For box counting dimension we know that

$$\liminf_{\varepsilon \to 0} N(A, \varepsilon)\, \varepsilon^s = \begin{cases} \infty & \text{if } 0 \le s \le \dim_B(A) \\ 0 & \text{if } \dim_B A < s < \infty \end{cases}.$$

The box counting dimension is also defined like this.

Proposition 3.4. Let A $\subset \mathbb{R}^n$ and let constants $c > 0$ and $\alpha > 0$ are given. Then if $f : A \to \mathbb{R}^m$ is a mapping for which $|f(x) - f(y)| \le c\, |x-y|^\alpha$ for all x , y $\in$ A. Then for each s, $H^{\frac{s}{\alpha}}(f(A)) \le c^{\frac{s}{\alpha}} H^s(A)$.

Proof. Let $\{ U_i \}$ be a δ - cover of A. Then $\{ f (A \cap U_i) \}$ is an ε - cover of f(A) since $|f(A \cap U_i)| \le c\, |U_i|^\alpha$ where $\varepsilon = c\, \delta^\alpha$. Then

$$\sum_i |f(A \cap U_i)|^{\frac{s}{\alpha}} \le c^{\frac{s}{\alpha}} \sum_i |U_i|^s$$

and so $H_\varepsilon^{\frac{s}{\alpha}}(f(A)) \le c^{\frac{s}{\alpha}} H_\delta^s(F)$.

Therefore, taking $\delta \to 0$, we have $H^{\frac{s}{\alpha}}(f(A)) \le c^{\frac{s}{\alpha}} H^s(A)$.

REFERENCES

[1] L.V. Ahlfors, Complex Analysis, McGraw-Hill New-York, (1979).

[2] M.F. Barnsley, Fractal Everywhere, Academic Press, (1988).

[3] ______ and S.G. Demcko, Iterated schemes and the global construction of fractals, Proc. R. Soc A399 (1985), 243 - 275.

[4] ______, Chaotic Dynamics and Fractals, Academic Press New York, (1986).

[5] A. F. Beardon , On the Hausdorff dimension of general Cantor sets, Proc. Camb. Phil. Soc. 61 (1965).

[6] M. V. Berry and Z. V. Lewis, On the Weierstrass-Mandelbrot fractal function, Proc. R. Soc. A370 (1980).

[7] A. S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets, (1928).

[8] ______, On fundamental geometric properties of plane line sets, J. London Math. Soc. 39 (1964) 441-448.

[9] F.M. Dekking, Recurrent sets, Adv. Math. 44 (1982), 78-104.

[10] R.L. Devaney, An introduction to chaotic dynamical systems, Benjamin cummings Publishing company, (1986).

[11] K.J. Falconer, The geometry of fractal sets, Cambridge Univ. Press Cambridge, (1985).

[12] ________, The Hausdorff dimension of distance sets, Mathematika 32 (1985c) 206-212.

[13] J. Feder, Fractals, Plenum Press New-York, (1988).

[14] A. Gerald, Edgar measure, topology and fractal geometry, Springer-Verlag New-York.

[15] J. E. Hutchinson, Fractals and self similarity, Indiana Univ. J. Math. 30 (1981) 713-747.

[16] S. A. Kline, On curves of fractal dimensions, J. London Math. Soc. 20 (1945) 79-86.

[17] B. B. Mandelbrot, Self-affine fractal sets, In fractals in Physico L. Pietroner and E. Tosatti, editors Elsevier Science Publ., (1986).

[18] Mauldin R. D. and William S. C., Random recursive constructions: asyptotic geometric and topological properties, Trans. Am. Math. Soc 295 (1986) 325-346.

[19] P. A.P. Moran, Additive functions of intervals and Hausdorff measure, Proc. Camb. Phil. Soc. 42 (1946) 15-23.

[20] C.A. Rogers, Hausdorff measures, Camb. Univ. Press, (1970).

[21] S.J. Taylor, The measure theory of random fractals, Math. Proc. Camb. Phil. Soc. 100 (1986) 383-406.

[22] Stringer, - Verlag, Fractals for the class room, Nation Council of teachers of Mathematics Advisory Board. (1992).