Mean Residual Life for the Modified Bathtub Curve

-修正壽命曲線의 平均殘存壽命-

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요 약

본 연구는 초기에 모집단에 취약한 부분에서 고장률의 증가가 생성하는 경우의 수정수명곡선에 관련한 것이다. 이후 고장률은 전형적 수명곡선에 따른다. 따라서 고장률 h(t)와 평균잔존수명 m(t)와의 관계는 새로운 비교가 제시되어야 하고 새로운 수리적 분석을 수행하여야 한다. 본 연구의 수리적 분석의 결과로 고장률과 평균잔존수명에 대한 최적Burn-In 타임은 일치하지 않는다.

Introduction

In practice, some industrial products (electronical components, mechanical products, and etc.) exhibit a "bathtub"curve for their failure rate functions as shown in Figure 1 when the products have weak parts and strong parts for their life. This curve is called a modified bathtub curve and is discussed by Jensen and Petersen [5] . The modified bathtub curve consists of four stages. The first stage has an increasing failure rate to some time, t(*). Thereafter, the failure rate follows the traditional "bathtub"curve. In most products, the early failures corresponds to a small percentage of the population. The useful region has a constant failure rate in which failure occur randomly. Many products exhibit this pattern. The wearout region has a increasing failure rate. Wearout occurs toward the end of product life when deterioration and weakening effects of accumulated stresses lead to an increasing failure rate function.

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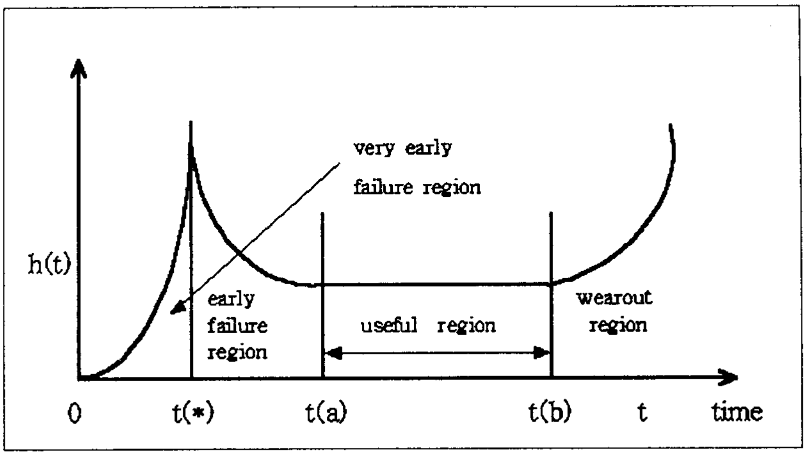

Figure 1. Modified Bathtub Curve, suggested by Jensen and Peterson.

Burn-in has been used to improve reliability measures such as failure rate ($h(t)$) and mean residual life ($m(t)$). Chandrasekaran [2] showed the case when the optimal burn-in time for mean residual life and failure rate are different. Then, Park [9] examined the effect of brun-in on the mean residual life and showed that the optimal burn-in times for failure rate and mean residual life are different for the traditional bathtub curve. For some cases, Guess and Park [3] showed that the burn-in times for minimizing failure rate and maximizing mean residual life are the same. Recently, Guess and Walker [4] discussed the optimal burn-in times for the three reliability measures. They showed by use of examples that the optimal burn-in times for the reliability measures are not the same. However, the relationship between failure rate and mean residual life has not been previously discussed under the modified bathtub curve. In this paper, the characteristics of these two measures will be examined assuming weak components are exist. Also, graphical data-analysis will be presented for studying the relationship between failure rate ($h(t)$) and mean residual life ($m(t)$).

Two-mixed Weibull distribution is a good model to describe the failure rate function of many components or systems if they have weak parts and main parts for stress. Kao [7] introduced a two-mixed Weibull distribution to describe the failure time of electronic tubes. Stitch [11] found that the failure time of microcircuits follow a mixed distribution law. Reynolds & Stevens [10] also found that two-mixed Weibull distribution describe the time-to-failure patterns of electronic components. For electro-mechanical device, Boardman and Colvert [1] found that the failure time of oral irrigators follow a two-mixed Weibull distribution. By focusing our attention on early failure for a component, interest necessarily narrows to the first three stages in the modified bathtub curve of Figure 1.

Two-Mixed Weibull Distribution

A two-mixed Weibull distribution is composed of two cumulative density function (CDF). Let $F_1(t)$ be the CDF of the weak population (small proportion of· susceptible components), $F_2(t)$ be the CDF of the main population (the rest of the components), and $F(t)$ be the total CDF for the entire population. Then, $F(t)$ is constructed by taking a weighted average of the CDFs for the weak and main subpopulation. The weights are the proportions of each type of subpopulation. Thus, if the weak population has p proportion, and the main population has $(1-p)$ proportion, then

$$F(t) = p\,F_1(t) + (1-p)\;F_2(t) \tag{1}$$

Typically, $F_1(t)$ has a high early failure rate while $F_2(t)$ has a low early failure rate that either stays constant or increases very late in life in the equation(1).

Assume that $f_i(t)$ is the probability density function (pdf) for $F_i(t)$ where $i=1,2$. Then, the failure rate of the two-mixed distribution is expressed as

$$h(t) = \frac{p\,f_1(t) + (1-p)\;f_2(t)}{1-\{p\,F_1(t) + (1-p)\;F_2(t)\}} \tag{2}$$

where $h(t)$ is interpreted as the instantaneous failure rate at time t.

Now, let us consider the two-mixed Weibull distribution with two parameters for each population. From the equation (1), the CDF of the two-mixed Weibull distribution is as follow:

$$F(t) = 1 - p(\exp([-(t/\ \eta_1)^{\ \beta_1}]) - (1-p)(\exp[-\ (t/\ \eta_2)^{\ \beta_2}]) \tag{3}$$

where

β_1 = shape parameter of the weak population,

β_2 = shape parameter of the main population

η_1 = scale parameter of the weak population,

η_2 = scale parameter of the main population

p = proportion of the weak population

When the shape parameter is less than one, we observe a decreasing failure rate function. When the shape parameter is equal to one, we get constant failure rate. When the shape parameter is greater than one, an increasing failure rate function results. Therefore, the shape parameter determines in which failure region a product belongs. The scale parameter is also called the characteristic life; the point at which 63.2% of units will have failed.

From equations (2) and (3), the failure density function, reliability function, and failure rate function can be expressed as follow:

$$f(t)=p\left(\frac{\beta_1}{\eta_1}\right)\left(\frac{t}{\eta_1}\right)^{\beta_1-1}\exp[-(t/\eta_1)^{\beta_1}]+(1-p)\left(\frac{\beta_2}{\eta_2}\right)\left(\frac{t}{\eta_2}\right)^{\beta_2-1}\exp[-(t/\eta_2)^{\beta_2}]$$

$$R(t)=p(\exp[-(t/\eta_1)^{\beta_1}])+(1-p)(\exp[-(t/\eta_2)^{\beta_2}])$$

$$h(t)=\frac{p\left(\frac{\beta_1}{\eta_1}\right)\left(\frac{t}{\eta_1}\right)^{\beta_1-1}\exp[-(t/\eta_1)^{\beta_1}]+(1-p)\left(\frac{\beta_2}{\eta_2}\right)\left(\frac{t}{\eta_2}\right)^{\beta_2-1}\exp[-(t/\eta_2)^{\beta_2}]}{p(\exp[-(t/\eta_1)^{\beta_1}])-(1-p)(\exp[-(t/\eta_2)^{\beta_2}])}$$

The two-mixed Weibull distribution will be used to compare the reliability measures.

Reliability Measures

Two reliability measures are usually considered for assessment during burn-in. These are failure rate ($h(t)$) and mean residual life ($m(t)$). Failure rate ($h(t)$) is interpreted as the instantaneous failure rate at time t. Mean residual life ($m(t)$) is the average life time remaining after burn-in to time t. Let X be the random life of a component, sub-system, or system with $f(x)$ its density. Then, the common definitions are:

mean residual life: $$m(t)=\int_t^\infty \frac{R(u)}{R(t)}\,du \quad (4)$$

failure rate: $$h(t)=\frac{f(t)}{R(t)}=-\frac{1}{R(t)}\left[\frac{d}{dt}R(t)\right] \quad (5)$$

where reliability: $R(x)=\int_x^\infty f(u)du=\exp\left[-\int_0^\infty h(u)du\right]$, $R(x)>0$ and $R(t)>0$.

Traditionally, burn-in is used to maximize the mean residual life of devices. Since we are interested in finding the burn-in time t at which $m(t)$ attains its maximum, let us study same properties of $m(t)$ which will be needed later. From equation (4), the first derivative of $m(t)$ is obtained as follows:

$$\begin{aligned} m'(t)&=\frac{\partial}{\partial t}\left[\frac{1}{R(t)}\int_t^\infty R(u)du\right]\\ &=h(t)\int_t^\infty \frac{R(u)}{R(t)}du-1\\ &=\frac{1}{R(t)}\left[h(t)\int_t^\infty R(u)du-R(t)\right] \end{aligned} \quad (6)$$

To simplify equation (6), let

$$L(t) = h(t)\int_t^{\infty} R(u)du - R(t). \tag{7}$$

Then, the first derivative of $m(t)$ can be expressed as follow:

$$m'(t) = \frac{1}{R(t)} L(t). \tag{8}$$

Thus, the first derivative of mean residual life depends on the sign of $L(t)$ since the reliability function is always greater than (or equal to) zero. In other words, if $L(t)>0$, then $m'(t)>0$ which implies that mean residual life ($m(t)$) is an increasing function.

$L(t)$ is a just function defined in equation (7) We introduce the function, $L(t)$, here to use it for examining the relation between mean residual life ($m(t)$) and failure rate ($h(t)$) under the modified bathtub curve. In equation (7), if $t \to \infty$, we have

$$L(t=\infty) = \lim_{t\to\infty}\left[h(t)\int_t^{\infty} R(u)du - R(t)\right] = 0 \tag{9}$$

Let us define the property of $L(t)$. From equation (7), the first derivative of $L(t)$ is

$$\begin{aligned} L'(t) &= \frac{\partial}{\partial t}\left[h(t)\int_t^{\infty} R(u)du - R(t)\right] \\ &= h'(t)\int_t^{\infty} R(u)du + h(t)(-R(t)) + f(t) \\ &= h'(t)\int_t^{\infty} R(u)du. \end{aligned} \tag{10}$$

From equation (10), we see that the sign of $L'(t)$ is the same as that of $h'(t)$. Thus, if $h'(t) > 0$, then $L'(t) > 0$ and so $L(t)$ is an increasing function. If $h'(t) < 0$, then $L'(t) < 0$ and so $L(t)$ is a decreasing function.

In the next section, we will define the relationship between mean residual life ($m(t)$) and failure rate ($h(t)$) under the modified bathtub curve using equations (8), (9), and (10).

Failure Rate and Mean Residual Life

Now, let us compare mean residual life, $m(t)$ and failure rate ($h(t)$) for the modified bathtub curve. Figure 2 presents the relationship between mean residual life and $h(t)$. We consider the curve in Figure 1 from right to left.

· In wearout region (between $t(b)$ and ∞ in Figure 1), the failure rate is increasing. Thus, $h(t) > 0$. This implies that $L'(t) > 0$ from equation (10) and so $L(t)$

(9). Then, we have $m'(t) < 0$ from equation (8). This implies that $m(t)$ is a decreasing function in wearout region.

· In the useful region (between $t(a)$ and $t(b)$ in Figure 1), the failure rate is constant. This implies that $h'(t)=0$. Then, we have $L'(t)=0$ from equation (10). Since $L'(t)=0$, $L(t)$ where $t\in[t(a), t(b)]$ is constant but smaller than 0 because $L(t(b)) < 0$. Then, we have $m'(t) < 0$ from equation (8). This implies that $m(t)$ is a decreasing function in the useful region.

· In the early failure region (between $t(*)$ and $t(a)$ in Figure 1), the failure rate is decreasing. Thus, $h'(t) < 0$ which implies that $L'(t) < 0$ by equation (10). Then, $L(t)$ where $t\in[t(*), t(a)]$ is a decreasing function. Since $L(t(a)) < 0$, it might be possible to have one solution of $L(t)=0$ or $L(t) < 0$.

First, consider the case when $L(t)=0$. Then, $m'(t)=0$ from equation (8). There is a t satisfying $m'(t)=0$ and $m''(t) < 0$ since $m''(t)=m(t)h'(t)+m'(t)h(t)$.
Thus, $m(t)$ is quasi-concave in this region and the relative maximum mean residual life $(m(t))$ lies between $t(*)$ and $t(a)$ in Figure 1.

For the second case when $L(t) < 0$, $m'(t) < 0$ from equation (8). Then, $m(t)$ is a decreasing function in the early failure region.

· In the very early failure region (between 0 and $t(*)$ in Figure 1), the failure rate is increasing. Thus, $h'(t) > 0$. This implies that $L'(t) < 0$ and so $L(t)$ where $t\in[0, t(*)]$ is increasing function Since $L(t(*))=0$ or $L(t(*)) < 0$, it might be possible to have one solution of $L(t)=0$ or $L(t) < 0$. Then, we have two possible cases.

For the first case when $L(t)=0$, $m'(t)=0$ from equation (8). There is a t satisfying $m'(t)=0$ and $m''(t) > 0$ since $m''(t)=m(t)h'(t)+m'(t)h(t)$. Thus. $m(t)$ is quasi-convex in this region. Then, the relative minimum $m(t)$ lies between 0 and $t(*)$.

For the second case $L(t) < 0$, $m'(t) < 0$ from equation (8). Then, $m(t)$ is a decreasing function in the very early failure region.

From the above results, we conclude that the burn-in times for optimum $m(t)$ and $h(t)$ are different. As shown in Figure 2, the maximum $m(t)$ is attained at
burn-in time between $t(*)$ and $t(a)$ (or at burn-in time $t=0$) while minimum failure rate is attained at the time $t(a)$ (or $t\in[t(a), t(b)]$). The relation between mean residual life $(m(t))$ and failure rate $(h(t))$ is presented in Figure 2.

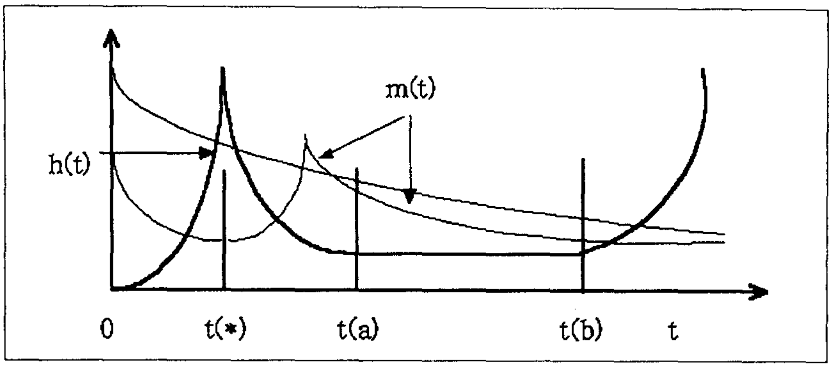

Figure 2. The Relationship between m(t) and h(t) under the Modified Bathtub Curve.

Note: If there is not one solution of $L(t)=0$ in the early failure region, the maximum $m(t)$ is attained at burn-in time $t=0$ and $m(t)$ is decreasing function.

An Illustrative Example

Life times for 19 components are recorded in Table 1 from $n=150$ CMOS components tested at $125^0 c$ and $5V$. This data was analyzed by Kececiloglu and Sun [8] to estimate the parameter values of mixed population. Here, our goal is to illustrate the characteristics of two reliability measures. i.e., the optimal burn-in times for $m(t)$ and $h(t)$ are not coincide each other.

Table 1. Time-to-failures in hours for CMOS components

100	200	250	420	420	588	588	588	708	1044
2892	2892	3396	3396	3997	3997	3997	4165	4500	

Applying Jensen and Petersen [5] method and Bayesian approach (Kamath) [6] to the data, Kececiloglu and Sun found that the time-to-failure pattern of the data follows two-mixed Weibull distribution law with the estimated parameter values as follow:

$$\hat{p}=0.067,\ \widehat{\beta_1}=1.62,\ \widehat{\eta_1}=535,\ \widehat{\beta_2}=4.3,\ \widehat{\eta_2}=8250.$$

The failure rate function of the data in Table 1 follows the modified bathtub curve. From the estimated parameter values. we estimate the reliability measures. Table 2 presents the estimated reliability measures. The optimal values and corresponding burn-in times for the reliability measures are determined from the Table 2 directly; i.e., find the optimal values of the reliability measures, then read the corresponding burn-in time t from the first column.

Table 2. The estimated values of $h(t)$ and $m(t)$ for the example problem

time t	*h(t) * 1000*	*m(t)*
0	0	2007
500	0.082725	2124
750	0.047194	2677
1000	0.020791	2427
1250	0.008076	2442
1750	0.003619	1942
2000	0.004957	1692

From Table 2, the optimal value for the failure rate ($h(t)*1000$) is the 0.003619. Thus, the corresponding burn-in time is the 1750 hours. For the mean residual life ($m(t)$), we have the optimum value at burn-in time 750 hours. Therefore, optimal burn-in times for the two reliability measures are different for this case.

Discussion

The relationship between two measures of reliability; namely failure rate ($h(t)$) and mean residual life ($m(t)$) has been discussed and mathematical graphical analyses are performed for the relationship under the modified bathtub curve. The results of mathematical and graphical analyses show that the burn-in time that optimizes one reliability measure does not yield an optimal value for another measure when a failure rate function follows the modified bathtub curve.

In practice, it is not unusual to have mixed population (weak population and main population) for many electrical or electro-mechanical products for their life. However, products have different reliability objectives. For example, failure rate ($h(t)$) will be a required objective for most CMOS components. For the electro-mechanical product such as an oral irrigator (Boardman and Colvert) [1] , the mean residual life ($m(t)$) is the objective to be maximized. For a process with weak population, we recommend that the first step of burn-in should be identifying the most appropriate reliability measure.

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