⊠연구논문

Designs and Comparison of Step and Constant-Stress ALTs for Acceleration Factor and Lognormal Lifetime Distributions *

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Abstract

This paper considers designing the simple (2-level) constant- and step-stress ALTs minimizing the asymptotic variance of the maximum likelihood estimator of the acceleration factor, which is defined as the ratio of the 100qth percentile at use stress to that a specified stress, for items having lognormally-distributed lives. It is assumed that (i) the log-linear relationship exists between the stress and the mean log life, (ii) the standard deviation of the log life is constant, and (iii) the cumulative exposure model holds for the effect of changing stress. For the constant-stress ALT the low stress and the sample proportion allocated to low stress are determined and for two modes of stress loading of step-stress ALTs, the low-to-high and high-to-low, the low stress and the stress change time are determined. For selected values of the design parameters the optimum plans are figured, two modes of step-stress ALTs and the constant-stress ALT are compared to each other, and the effects of the incorrect pre-estimates of the design parameters are investigated.

1. Introduction

When life testing of items at the specified use condition requires a long time to acquire the test data, accelerated life tests (ALTs) or partially accelerated life tests (PALTs) are often used to shorten the lives of test items. In an ALT test items are run only at higher-than-usual levels of stress, and in a PALT at both

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accelerated and use stresses. The test data obtained at the accelerated conditions are analyzed in terms of a model, and then extrapolated to the specified design stress to estimate the life distribution.

For constant-stress ALTs where the test condition of an item is constant over time the monograph [Nelson, 1990] gives various methods of estimating the parameters involved. When lifetimes of items follow a Weibull or lognormal distribution, optimal constant stress ALT plans which minimize the asymptotic variance of maximum likelihood estimator (MLE) of a specified percentile at use condition are available in the literature; see monographs [10, 12], and references therein. A step stress ALT allows the test condition to change at a given time or upon the occurrence of a specified number of failures. The former is called the time-step stress test and the latter failure-step stress test. Nelson(1990) presented a statistical model for analyzing the data from a step stress test. For items having exponentially distributed lives [4, 5, 11, 12] the optimum simple step-stress ALTs gave the stress change time and the proportion of failures at low stress which minimize the asymptotic variance of MLE of the log mean life at design stress. Here the word 'simple' means that only two stress levels, low and high, are used in a test. For items whose lives follow a Weibull distribution Schatzoff and Lane(1987) studied the problem with readout data. Bai and Kim(1983) presented simple time-step stress ALTs with Type I censoring which minimize the asymptotic variance of MLE of a specified percentile at design stress, and found optimal plans.

DeGroot and Goel(1979) considered a PALT in which a test item is first run at use condition and, if it does not fail for a specific time τ, then it is run at accelerated condition until failure. They suggested the model in which $Y = T$ if $T \leq \tau$, and $Y + \tau + (T - \tau) / \beta$ if $T > \tau$, where T is the lifetime of an item at use condition and Y its total lifetime. Assuming that T follows an exponential distribution with mean life α and using a Bayesian approach, they obtained estimators of β and α, and the optimal change time τ^*. Bhattacharyya and Soejoeti(1989) proposed a failure rate model in which, if $h_T(\cdot)$ and $h_Y(\cdot)$ are the failure functions of T and Y, respectively, then $h_Y(y) = h_T(y)$ if $y \leq \tau$ and $h_Y(y) = \beta h_T(y)$ if $y > \tau$. This model is equivalent to the DeGroot-Goel model when T has an exponential distribution.

An optimally designed PALT can have some practical usage for problems where one has to know the acceleration factor in order to carry out the test only at specified acceleraedted consdition and to extrapolate the data to estimate the lifetime distribution at use condition. See DeGroot and Goel(1979), and Bhattacharyya and Soejoeti(1989). The problem of optimally designing PALTs has been considered by DeGroot and Goel(1979) and Bai and Chung(1992) for items with exponentially-distributed lives, and by Bai et al.(1993) for items having lognormally-distributed lives. For the problems where the relationships relating test conditions to mean lives of items (log-linear, log-quadratic, etc) are known except for a few parameters, it is possible to estimate the acceleration factor from ALT data. In such situations ALTs can be more effective than PALTs. However, if such relationships are not known or cannot be assumed, ALT data cannot be extrapolated to use condition, so PALTs are the reasonalbel schemes to adopt.

This paper considers the optimal designs of the simple constant- and step-stress ALTs for the life tesing problems where (i) each item has a lognormally-distributed lifetime, (ii) the log-linear relationship exists between the stress and the mean loglife, (iii) the standard deviation of the log life is constant, and (iv) the cumulative exposure model proposed by Nelson(1990), and Miller and Nelson(1983) holds for the effect of changing stress. The model parameters including the acceleration factor, which is defined as the ratio of the 100qth percentile at use (reference) stress to that at a specified stress [Nelson, 1990] are to be estimated by method of maximum likelihood. For a constant-stress ALT the low stress and the sample proportion allocated to low stress are determined to minimize the asymptotic variance of MLE of the acceleration factor, and for the two modes of step-stress ALTs the low stress and the stress chane time. For selected values of the design parameters the optimum plans are figured, two modes of step-stress ALTs and the constant-stress ALT are compared to each other, and the effects of the incorrect pre-estimates of the design parameters are investigated.

2. The Model

Notations

n, η	total number of test items and censoring time
τ_L, τ_H, π, x_L, x_L	stress change times in low-to-high and high-to-low modes, and proportion allocated to low stress in a constant-stress test; $x_L = \tau_L / \eta$; $x_H = \tau_H / \eta$
ξ_0, ξ_1, ξ_2, s	design, low, and high stresses; $s = s(\xi) = (\xi - \xi_0)/(\xi_2 - \xi_0)$
Y	total lifetime of an item in a step-stress test

μ, σ location and scale parameters of the lognormal lifetime distribution

α_0, α_1 (or γ_0, γ_1) parameters of the log-linear relation between μ and stress

$\phi(\cdot)$, $\Phi(\cdot)$, $\Phi^{-1}(\cdot)$ standard normal p.d.f.c., c.d.f., inserse function of $\Phi(\cdot)$

p_1, p_2 probabilities that an item will fail at the low and high stresses;

p_d, p_h, a, b, p_s probabilities that an item tested only at design and high stresses will fail by η; $a = \Phi^{-1}(p_h)$, $b = \Phi^{-1}(p_h) - \Phi^{-1}(p_d)$; $p_s = 1 - p_1 - p_2$

Assumptions

1. At any constant stress ξ, the lifetime of an item follows a lognormal distribution with location parameter $\mu(\xi) = \alpha_0 + \alpha_1 \xi$ and scale parameter σ.
2. The lognormal scale parameter σ is independent of stress.
3. The cumulative exposure model [Nelson, 1990] holds for the effect of changing stress.
4. The lifetimes of test items are stochastically independent.

Standardized model

Define the standardized stress as

$$s = (\xi - \xi_0) / (\xi_2 - \xi_0), \tag{2.1}$$

where ξ_0 and ξ_2 are the prespecified design and high stresses, respectively. For the high stress ξ_2, $s_2=1$, and for the design stress ξ_0, $s_0 = 0$. The location parameter of the lognormal distribution can be rewritten in terms of the standardized stress s as $\mu(s) = \gamma_0 + \gamma_1 s$, where $\gamma_0 = \mu(\xi_0)$, and $\gamma_1 = \mu(\xi_2) - \mu(\xi_0)$.

Acceleration factor

The acceleration factor $\beta(\xi)$ between stress ξ and use stress ξ_0 is defined as ratio of the 100qth percentile at use stress ξ_0 to that at stress ξ and is

$$\beta(\xi) = \frac{e^{\mu(\xi_0)+z_q\sigma}}{e^{\mu(\xi)+z_q\sigma}} = e^{-\alpha_1(\xi-\xi_0)}, \tag{2.2}$$

where z_q is the 100qth percentile of a standard normal distribution, and in terms of standardized stress it is written as

$$\beta(s) = e^{-\gamma_1 s}. \tag{2.3}$$

2.1 Low-to-high Step-stress ALTs

Test procedure

1. Each of n test items is first run at low stress s_1 (or ξ_1).
2. If it does not fail at low stress s_1 by τ_1, then it is put to high stress s_2(= 1or ξ_2) and run until censoring time η.

Lifetime distribution

Let Y be the random variable representing the total lifetime of an item under a simple low-to-high mode test. Also let $z_1(y) = (\ln y - \gamma_0 - \gamma_1 s_1)/\sigma$, and $z_2(y) = (\ln(y - \tau_L + \delta_L) - \gamma_0 - \gamma_1))/\sigma$, where $\delta_L = \tau_L \exp(\gamma_1(1 - s_1))$. From the above assumptions, the p.d.f. of Y is

$$f(y) = \begin{cases} 0, & y \le 0 \\ \phi(z_1(y)) \ / \ (\sigma y), & 0 < y \le \tau_L. \\ \phi(z_2(y)) \ / \ (\sigma(y - \tau_L + \delta_L)), & \tau_L < y \end{cases} \tag{2.4}$$

Estimation of parameters

The method of maximum likelihood is used to estimate parameters γ_0, γ_1 and σ from the test data. The low stress s_1 and stress change time τ_L are determined to minimize the asymptotic variance of MLE of the log accleration factor, $-s\gamma_1$. The lifetimes $Y_1, \cdots, Y_n$ of n test items are independent and identically distributed random variables. Therefore it is sufficient to consider the likelihood for a single observation. Let y_i be the observed value of the total lifetime of item i, and $D_1 = \{y_i \mid 0 < y_i \le \tau_L\}$, and $D_2 = \{y_i \mid \tau_L < y_i \le \eta\}$. And let indicator functions I_{ij}'s be defined as

$$I_{ij} = I_{ij}(y_i) = \begin{cases} 1, & y_i \in D_j \\ 0, & y_i \notin D_j \end{cases}, \quad i = 1, 2, \cdots, n,\ j = 1, 2. \tag{2.5}$$

The log likelihood for (y_i, I_{i1}, I_{i2}) is

$$\begin{aligned} l_i &= I_{i1}\left\{-\ln\sigma - \ln y_i - \frac{1}{2}z_1^2(y_i) - \frac{1}{2}\ln(2\pi)\right\} \\ &\quad + I_{i2}\left\{-\ln\sigma - \ln(y_i - \tau_1 + \delta_1) - \frac{1}{2}z_2^2(y_i) - \frac{1}{2}\ln(2\pi)\right\} + I_{i3}\ln\overline{\Phi}(\varphi), \end{aligned} \tag{2.6}$$

where $\overline{\Phi}(\cdot) = 1 - \Phi(\cdot)$, $I_{i3} = (1 - I_{i1} - I_{i2})$, and $\varphi = z_2(\eta)$. The log likelihood for $(y_i, I_{i1}, I_{i2}, i = 1, \cdots, n)$ is $l(\gamma_0, \gamma_1, \sigma) \equiv \sum_{i=1}^{n} l_i$. MLEs $\widehat{\gamma_0}$, $\widehat{\gamma_1}$ and $\hat{\sigma}$ are the values of γ_0, γ_1 and σ which are solutions to the system of equations obtained by letting the first partial derivatives of the total log likelihood $l(\gamma_0, \gamma_1, \sigma)$ with respect to γ_0, γ_1 and σ be zero. If we let $z_3(y_i) = -\sigma(\partial z_2(y_i)/\partial\gamma_1) = 1 - (1 - s_1)\delta_L/(y_i - \tau_L + \delta_L)$, and $z_4(y_i) = 1 - z_3(y_i)$, then the system of equations is as follows:

$$\sigma(\partial l/\partial\gamma_0) = \sum_{i=1}^{n}[I_{i1}z_1(y_i) + I_{i2}z_2(y_i) + I_{i3}\lambda(\varphi)] = 0, \tag{2.7}$$

$$\sigma(\partial l/\partial\gamma_1) = \sum_{i=1}^{n}[s_1 I_{i1} z_1(y_i) + I_{i2}\{-\sigma z_4(y_i) + z_2(y_i)z_3(y_i)\} + I_{i3}\rho\lambda(\varphi)] = 0, \tag{2.8}$$

$$\sigma(\partial l/\partial\sigma) = \sum_{i=1}^{n}[I_{i1}\{z_1^2(y_i) - 1\} + I_{i2}\{z_2^2(y_i) - 1\} + I_{i3}\varphi\lambda(\varphi)] = 0, \tag{2.9}$$

where $\lambda(\varphi) = \phi(\varphi)/\overline{\Phi}(\varphi)$ and $\rho = -\sigma\varphi' = -\sigma(\partial\varphi/\partial\gamma_1) = z_3(\eta)$.

Fisher information matrix

The Fisher information matrix $F_n(\gamma_0, \gamma_1, \sigma)$ is obtained by taking expectations of the negative of the second partial derivatives of l with respect to γ_0, γ_1 and σ. Using $E(\partial l/\partial\gamma_0) = E(\partial l/\partial\gamma_1) = E(\partial l/\partial\sigma) = 0$, the second partial derivatives and their expectations (see the appendix), we obtain the following Fisher information matrix:

$$F_n(\gamma_0, \gamma_1, \sigma) = \frac{n}{\sigma^2}\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}, \tag{2.10}$$

where $f_{11} = -E(\partial^2 l/\partial\gamma_0^2) = 1 - p_s + p_s\lambda'(\varphi)$,

$f_{12} = -E(\partial^2 l/\partial\gamma_0\partial\gamma_1) = s_1p_1 + p_2 - (1 - s_1)g_1 + p_s\rho\lambda'(\varphi)$,

$f_{13} = -E(\partial^2 l/\partial\gamma_0\partial\sigma) = p_s\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\}$,

$f_{22} = -E(\partial^2 l/\partial\gamma_1^2) = s_1^2p_1 + p_2 - 2(1 - s_1)g_1 + (1 - s_1)^2(1 + \sigma^2)g_2 -$

$\sigma(1 - s_1)^2\phi(\varphi)e^{(\zeta - \varphi)\sigma}\{1 - e^{(\zeta - \varphi)\sigma}\} + p_s\{\sigma\omega\lambda(\varphi) + \rho^2\lambda'(\varphi)\}$,

$f_{23} = -E(\partial^2 l/\partial\gamma_1\partial\sigma) = 2\sigma(1 - s_1)g_1 + p_s\rho\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\}$,

$f_{33} = -E(\partial^2 l/\partial\sigma^2) = 2(1 - p_s) + p_s\varphi\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\}$,

$\zeta = z_1(\tau_L) = (\ln\tau_1 - \gamma_0 - \gamma_1 s_1)/\sigma$, $p_1 = \Phi(\zeta)$, $p_2 = \Phi(\varphi) - \Phi(\zeta)$,

$p_s = 1 - p_1 - p_2$, $\omega = \sigma(\partial^2\varphi/\partial\gamma_1^2) = (1 - s_1)^2(\eta - \tau_L)\delta_L /$

$(\eta - \tau_L + \delta_L)^2$, $g_i = e^{\zeta i\sigma + (i\sigma)^2/2}\{\Phi(\varphi + i\sigma) - \Phi(\zeta + i\sigma)\}$, $i = 1, 2$,

and $\lambda'(\varphi) = \partial\lambda(\varphi)/\partial\varphi = \lambda^2(\varphi) - \varphi\lambda(\varphi)$.

Case without censoring

For the case without censoring the elements of Fisher information matrix (2.9) are replaced by their limits as $\lim_{\eta\to\infty} p_2 = 1 - p_1$ $\lim_{\eta\to\infty} p_s = 0$, and $\lim_{\eta\to\infty} g_i = e^{\zeta i\sigma + (i\sigma)^2/2}$ $\overline{\Phi}(\zeta + i\sigma) \equiv g_i^{\infty}$, $i = 1, 2$.

$$F_n^{\infty}(\gamma_0, \gamma_1, \sigma) = \frac{n}{\sigma^2}\begin{bmatrix} 1 & f_{12}^{\infty} & 0 \\ f_{12}^{\infty} & f_{22}^{\infty} & f_{23}^{\infty} \\ 0 & f_{23}^{\infty} & 2 \end{bmatrix}, \tag{2.11}$$

where $f_{12}^{\infty} = 1 - (1 - s_1)(p_1 + g_1^{\infty})$,

$f_{22}^{\infty} = 1 - (1 - s_1^2)p_1 - 2(1 - s_1)g_1^{\infty} + (1 - s_1)^2(1 + \sigma^2)g_2^{\infty}$, and

$f_{23}^{\infty} = 2\sigma(1 - s_1)g_1^{\infty}$.

Asymptotic variance of MLE of the acceleration factor

The asymptotic variance-covariance matrix of MLEs $\widehat{\gamma}_0$, $\widehat{\gamma}_1$ and $\hat{\sigma}$ is the inverse matrix of $F_n(\gamma_0, \gamma_1, \sigma)$. MLE of the log acceleration factor $\widehat{\beta(s)}$ at stress s is

$$\ln \widehat{\beta(s)} = -s\,\widehat{\gamma}_1. \tag{2.12}$$

The asymptotic variance of the estimator $\ln \widehat{\beta(s)}$ is

$$Asvar(\ln \widehat{\beta(s)}) = s^2 Asvar(\widehat{\gamma}_1), \tag{2.13}$$

and the asymptotic variance of MLE of the acceleration factor at stress s is $s^2\beta^2(s)\, Asvar(\widehat{\gamma}_1)$. For the low-to-high mode without censoring the determinant of the Fisher information matrix is

$$|F_n^{\infty}(\gamma_0, \gamma_1, \sigma)| = 2(n^3/\sigma^6)(1-s_1)^2\{p_1 + (1+\sigma^2)g_2^{\infty} - 2\sigma^2(g_1^{\infty})^2, \\ -(p_1 + g_1^{\infty})^2\} \tag{2.14}$$

and

$$Asvar(\widehat{\gamma}_1) = 2\left(\frac{n}{\sigma^2}\right)^2 |F_n^{\infty}(\gamma_0, \gamma_1, \sigma)|^{-1}. \tag{2.15}$$

2.2 High-to-low Step-stress ALTs

Test procedure

1. Each of n items is first tested at high stress s_2(= 1 or ξ_2).
2. If it does not fail at high stress s_2 by τ_H, then it is put to low stress s_1(= 1 or ξ_1) and run until censoring time η.

Fisher information matrix

Let $z_1(\tau_H) = (\ln \tau_H - \gamma_0 - \gamma_1)/\sigma$, $\delta_H = \tau_H \exp(\gamma_1(s_1 - 1))$, and $z_2(\eta)$ $(\ln(\eta - \tau_H + \delta_H) - \gamma_0 - \gamma_1 s_1))/\sigma$. The Fisher information matrix for the high-to-low test is the matrix obtained by replacing the elements of matrix (2.10) with the following quantities (see the appendix):

$$f_{11} = 1 - p_s + p_s\lambda'(\varphi),$$
$$f_{12} = p_2 + s_1p_1 - (s_1 - 1)g_1 + p_s\rho\lambda'(\varphi),$$
$$f_{13} = p_s\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\},$$
$$f_{22} = s_1^2p_1 + p_2 - 2s_1(s_1 - 1)g_1 + (s_1 - 1)^2(1 + \sigma^2)g_2 -$$
$$\sigma(s_1 - 1)^2\phi(\varphi)e^{(\zeta - \varphi)\sigma}\{1 - e^{(\zeta - \varphi)\sigma}\} + p_s\{\sigma\omega\lambda(\varphi) + \rho^2\lambda'(\varphi)\},$$
$$f_{23} = 2\sigma(s_1 - 1)g_1 + p_s\rho\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\},$$
$$f_{33} = 2(1 - p_s) + p_s\varphi\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\},$$
$$\zeta = z_1(\tau_2),\ \varphi = z_2(\eta),\ p_2 = \Phi(\zeta),\ p_1 = \Phi(\varphi) - \Phi(\zeta),$$
$$p_s = 1 - p_1 - p_2,\ \rho = -\sigma(\partial\varphi/\partial\gamma_1) = (s_1(\eta - \tau_H) + \delta_H)/\ (\eta - \tau_H + \delta_H),$$
$$\omega = \sigma(\partial^2\varphi/\partial\gamma_1^2) = (s_1 - 1)^2(\eta - \tau_H)\delta_H\ /\ (\eta - \tau_H + \delta_H)^2,$$
$$g_i = e^{\zeta i\sigma + (i\sigma)^2/2}\{\Phi(\varphi + i\sigma) - \Phi(\zeta + i\sigma)\},\ i = 1,\ 2$$
$$\lambda'(\varphi) = \lambda^2(\varphi) - \varphi\lambda(\varphi).$$

For the high-to-low mode without censoring the Fisher information matrix is obtained by replacing the elements of matrix (2.11) with the following quantities:

$$f_{12}^{\infty} = s_1 - (s_1 - 1)(p_2 + g_1^{\infty}),$$
$$f_{22}^{\infty} = s_1^2 - (s_1^2 - 1)p_2 - 2s_1(s_1 - 1)g_1^{\infty} + (s_1 - 1)^2(1 + \sigma^2)g_2^{\infty},\ \text{and}$$
$$f_{23}^{\infty} = 2\sigma(s_1 - 1)g_1^{\infty},$$

where $p_2 = \Phi(\zeta)$, and $p_1 = 1 - p_2$. For the high-to-low mode without censoring the determinant of the Fisher information matrix and $Asvar(\hat{\gamma}_1)$ are same as those for the low-to-high mode.

2.3 Constant-stress ALTs

Test procedure

1. $n\pi$ items chosen randomly among n test items are allocated to high stress and the remaining $n(1 - \pi)$ items to low stress s_1.
2. Each test item is run until censoring time η or until failure.

Fisher information matrix

The Fisher information matrix, which either can be obtained by letting x_L=1.0 and n=1 in (2.10) or can be found in [Nelson, 1990], for one test item allocated to stress s_1 is

$$F_1(\gamma_0, \gamma_1, \sigma) = \frac{1}{\sigma^2}\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}, \qquad (2.16)$$

$$f_{11} = 1 - p_s + p_s\lambda'(\varphi), \quad f_{12} = s_1 f_{11}, \quad f_{13} = p_s\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\},$$

$$f_{22} = s_1^2 f_{11}, \qquad f_{23} = s_1 f_{13}, \quad f_{33} = 2p + \varphi f_{13},$$

$$\varphi = (\ln\eta - \gamma_0 - \gamma_1 s_1)/\sigma, \ p = \Phi(\varphi), \ p_s = 1 - p, \ \lambda'(\varphi) = \lambda^2(\varphi) - \varphi\lambda(\varphi),$$

and the Fisher information matrix for the n test items is $F_n(\gamma_0, \gamma_1, \sigma) = n\pi F_1(\gamma_0, \gamma_1, \sigma) + n(1-\pi)F_1(\gamma_0, \gamma_1, \sigma)$. For the constant-stress tesr without censoring the determinant of the Fisher information matrix is

$$|F_n^{\infty}(\gamma_0, \gamma_1, \sigma)| = 2(n^3/\sigma^6)(1 - s_1)^2\left\{-\left(\pi - \frac{1}{2}\right)^2 + \frac{1}{4}\right\}, \qquad (2.17)$$

and

$$Asvar(\widehat{\gamma_1}) = 2\left(\frac{n}{\sigma^2}\right)^2 |F_n^{\infty}(\gamma_0, \gamma_1, \sigma)|^{-1}. \qquad (2.18)$$

3. Optimum Plans

Design parameters

The optimal change time of a simple step-stress ALT depends on model parameters γ_0, γ_1 and σ. A design using the pre-estimates of the unknown model parameters is called a *locally* optimal design [7, 8] and is commonly adopted [2, 3, 5, 7, 8, 10, 12]. Let p_d and p_h be, respectively, the probabilities that an item tested only at design and high stresses will fail by censoring time η. Then we have $p_d = \Phi((\ln\eta - \gamma_0)/\sigma)$, and $p_h = \Phi((\ln\eta - \gamma_0 - \gamma_1)/\sigma)$. Let

$a = \Phi^{-1}(p_h) = (\ln\eta - \gamma_0 - \gamma_1)/\sigma$, and $b = \Phi^{-1}(p_h) - \Phi^{-1}(p_d) = -\gamma_1/\sigma$, where $\Phi^{-1}(\cdot)$ is the inverse function of the standard normal cumulative distribution function.

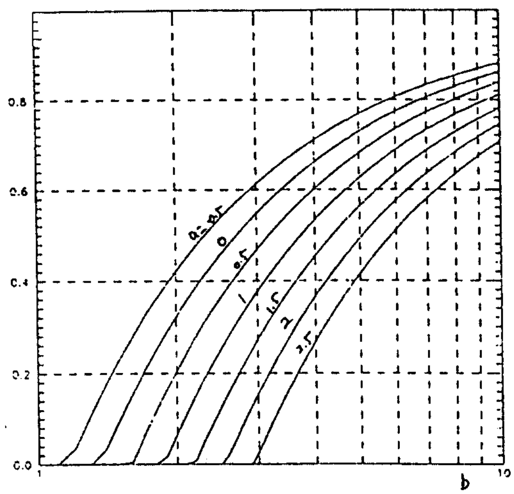
A step-stress test can be designed using the pre-estimate of either one of the three sets $(\gamma_0, \gamma_1, \sigma)$, (p_d, p_h, σ), or (a, b, σ). Let $x_L = \tau_L/\eta$ and $x_H = \tau_H/\eta$. It can be easily shown (see Appendix) that, for the low-to-high mode, the Fisher information matrix and the asymptotic variance of MLE of the log acceleration factor at stress s can be written in terms of s_1, x_L and either of $(\gamma_0, \gamma_1, \sigma)$, (p_d, p_h, σ), or (a, b, σ), and, for the high-to-low mode, they can be written in terms of s_1, x_H and either of $(\gamma_0, \gamma_1, \sigma)$, (p_d, p_h, σ), or (a, b, σ). However, the optimal asymptotic variance of MLE of the log acceleration factor at stress s for a constant-stress test is a function of s_1, π, and (p_d, p_h), or (a, b) [Nelson, 1990].

Optimal plans

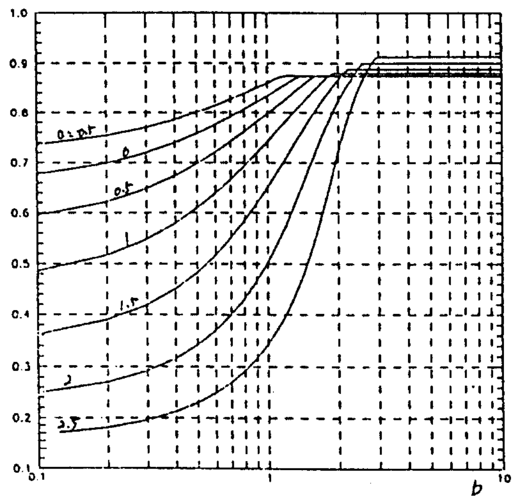
The optimal low stress s_1^* and low-to-high or high-to-low stress change times x_L^* or x_H^* for a simple step-stress test, and the the optimal low stress s_1^* and sample proportion allocated to low stress π^* for a constant-stress test can be obtained numerically, for example, by the Powell(1964) method. Note that the failure rate function $\lambda_T(t)$ of the lifetime T of an item at design stress is equal to $(\sigma t)^{-1}\phi((\ln t - \mu)/\sigma)/\overline{\Phi}((\ln t - \mu)/\sigma)$, and that over much of the distribution i) $\lambda_T(t)$ is roughly constant for $\sigma \cong 0.5$, ii) $\lambda_T(t)$ increases for $\sigma \leq 0.2$, and iii) $\lambda_T(t)$ increases quickly and decreases slowly for $\sigma > 0.8$ [Nelson(1990), pp. 62].

We have computed the optimal low stress and stress change times of two mode for various combinations of $(a, b, \sigma = 0.8)$ values. Figures 1-A, 1-B, 2-A, and 2-B show s_1^* and x_L^* or $1 - x_H^*$ for a = -0.5(0.5)2.5 as a function of b, and Figures 3-A and 3-B give s_1^* and π^*. Ratios of optimal $n\,Asvar(\widehat{\gamma}_1)$ for the low-to-high mode to that for the high-to-low mode, and ratios of $n\,Asvar(\widehat{\gamma}_1)$ for the constant-stress test to that for the high-to-low mode are shown Figures 4-A and 4-B. Note that, for combinations considered, the optimal $n\,Asvar(\widehat{\gamma}_1)$ for the high-to-low mode is smaller than that for the high-to-low mode, and the optimal $n\,Asvar(\widehat{\gamma}_1)$ for the constant-stress test is smaller than that for high-to-low mode. As an example suppose that the pre-estimate of (a, b, σ) is equal to (2.0, 4.0, 0.8), i.e., (p_d, p_h) = (0.97725, 0.02275). Then (s_1^*, x_L^*) = (0.36, 0.90) and

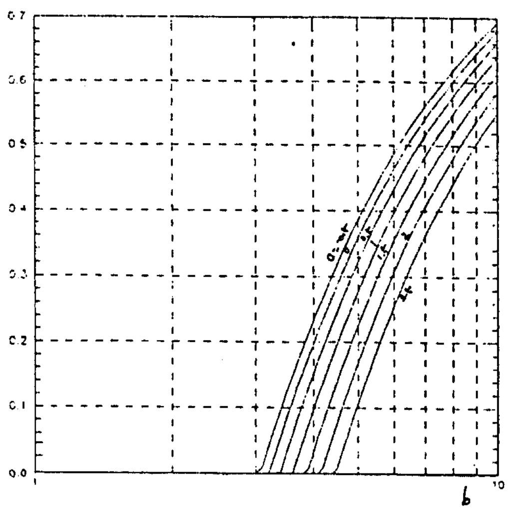
$(s_l^*, 1-x_H^*)$ = (0.0, 0.86) for the low-to-high and high-to-low modes, and (s_l^*, π^*) = (0.27, 0.54) for the constant-stress test from Figures 1-A ~ 3-B. From the eqaution of (2.10), and Figures 4-A and 4-B the optimal $Asvar(\widehat{\gamma}_1)$ = 27.5/n and the optimal $Asvar(\widehat{\gamma}_1)$ = 27.5/(1.90n) = 14.5/n for the low-to-high and the high-to-low modes, and the optimal $Asvar(\widehat{\gamma}_1)$ = 14.5/(1.71n) = 8.5/n for the constant-stress test.



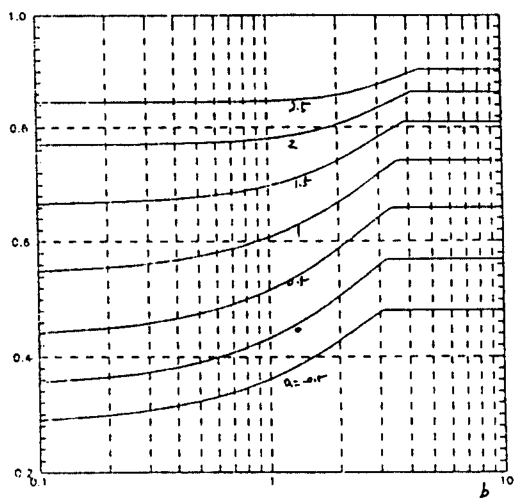
<Figure 1-A> s_l^* for low-to-high mode; σ=0.8



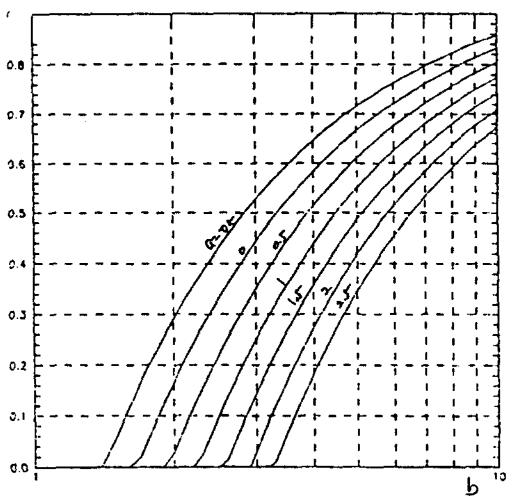
<Figure 1-B> x_L^* for low-to-high mode; σ=0.8



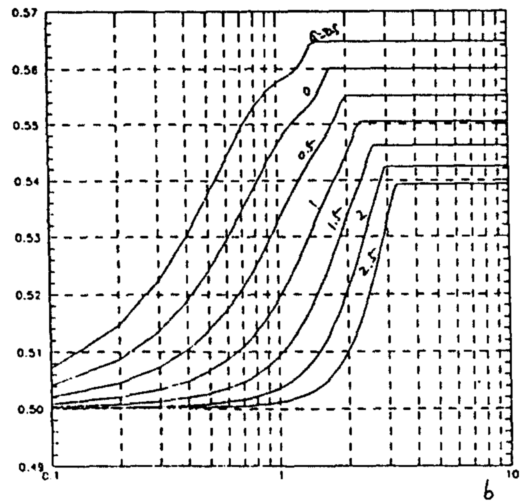
<Figure 2-A> s_l^* for high-to-low mode; σ=0.8



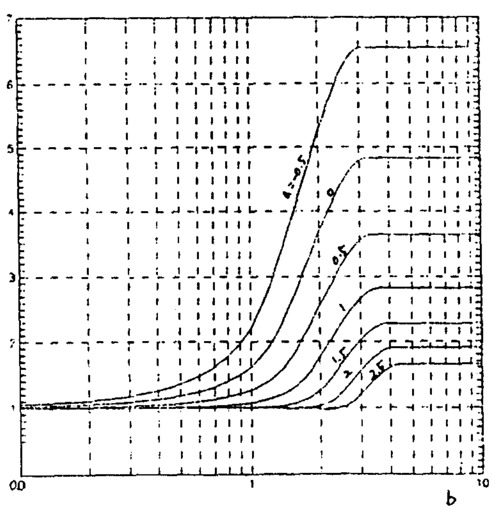
<Figure 2-B> $1-x_H^*$ for high-to-low mode; σ=0.8



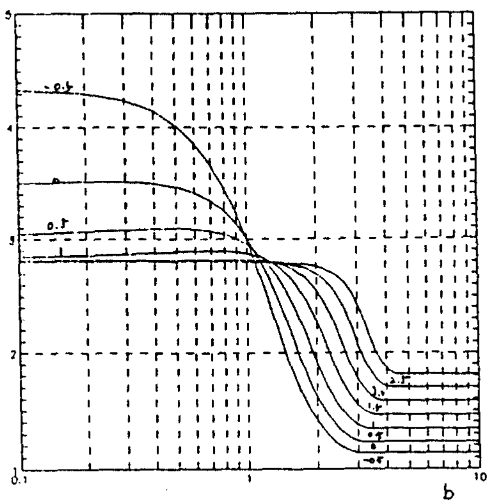
<Figure 3-A> s_1^* for constant-stress test; σ=0.8



<Figure 3-B> π^* for constant-stress mode; σ=0.8



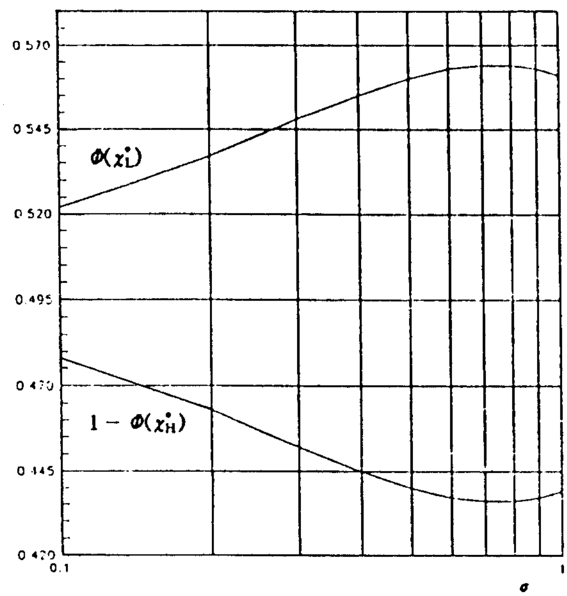
<Figure 4-A> Ratios of $\mathrm{n}\,Asvar^*(\widehat{\gamma}_1)$ for two modes



<Figure 4-B> Ratios of optimal $\mathrm{n}\,Asvar(\widehat{\gamma}_1)$ for high-to-low mode to that for constant-stress test

Case without censoring

The optimal $Asvar(\widehat{\gamma_1})$ for the low-to-high mode depends on s_1, γ_0, γ_1, σ and τ_L only through s_1, σ and s_1, σ, and $\chi_L = (\ln \tau_L - \gamma_0 - \gamma_1 s_1)/\sigma$ or equivalently s_1, σ and $\Phi(\chi_L) = \Phi((\ln \tau_L - \gamma_0 - \gamma_1 s_1)/\sigma)$. Similarily the optimal $Asvar(\widehat{\gamma_1})$ for the high-to-low mode depends on s_1, σ and $\Phi(\chi_H) = \Phi((\ln \tau_H - \gamma_0 - \gamma_1 s_1)/\sigma)$. Optimal $s_1^*=0$ for the step-stress test, and Figure 5 gives $\Phi(\chi_L^*) = \Phi((\ln \tau_L^* - \gamma_0 - \gamma_1 s_1)/\sigma)$ and $1 - \Phi(\chi_H^*) = 1 - \Phi((\ln \tau_H^* - \gamma_0 - \gamma_1 s_1)/\sigma)$ as a function of σ. Note that, as to the case where the low stress level is given, Figure 5 can be used for finding the optimal stress chage times. For the constant-stress test $s_1^*=0$, and $\pi^*=0.5$ from (2.15) and (2.16). As an example we assume that the pre-estimate of (γ_0, σ) is equal to (5, 0.8). Then from Figure 5 $\tau_L^* = \exp(5 + 0.8 \cdot \Phi^{-1}(0.5615)) = 167.98$ and $\tau_H^* = \exp(5 + 0.8 \cdot \Phi^{-1}(1 - 0.4350)) = 169.17$ from <Figure 5>.



<Figure 5> $\Phi(\chi_L^*)$ and $1 - \Phi(\chi_H^*)$ as a function of σ

4. Effects of Incorrect Pre-estimates

An optimal design of an ALT is determined by specifying the value of (a, b, σ) or (a, b) which is usually unknown. Therefore, the value of (a, b, σ) has to be pre-estimated from past experience, data for example similar item, or a preliminary test. Incorrect pre-estimates give a non-optimal test. We investigate the effects of the incorrect pre-estimate of (a, b, σ) in terms of the percentage of the asymptotic variance increase (PAVI). For the example of (a, b, σ) = (2.0, 4.0, 0.8) considered in Section 3, Table 1 shows PAVI due to using the incorrect pre-estimate (a', b', σ') of (a, b, σ) for the test minimizing the asymptotic variance of MLE of the log acceleration factor at stress s. Note that when correct pre-estimate of (a, σ) PAVI for the high-to-low mode is smaller than PAVIs for the other tests.

< Table 1 > PAVIs for step and constant-stress tests

	σ' = 0.7			σ' = 0.8			σ' = 0.9		
b'	1.75	2.0	2.25	1.75	2.0	2.25	1.75	2.0	2.25
3.0	34.6[1)]	66.6	112.5	35.3	68.9	119.9	34.9	69.7	125.4
	41.8[2)]	23.1	10.5	11.6	3.3	.1	.3	1.2	8.1
	22.0[3)]	41.4	56.5	22.0	41.4	56.5	22.0	41.4	56.5
3.5	3.0	12.1	29.4	2.7	11.7	29.1	2.3	10.9	28.0
	31.5	16.2	6.3	6.0	.8	.5	.3	5.3	17.4
	1.8	6.6	15.0	1.8	6.6	15.0	1.8	6.6	15.0
4.0	2.6	.1	2.	2.5	.0	2.6	2.9	.2	2.5
	23.6	10.8	3.3	2.5	.0	2.6	4.7	12.9	30.9
	1.2	.0	1.2	1.2	.0	1.2	1.2	.0	1.2
4.5	17.7	7.0	1.3	18.1	7.5	1.7	19.6	8.9	3.0
	18.1	7.0	1.4	5.4	3.3	7.4	16.7	27.9	51.3
	9.4	4.2	1.1	9.4	4.2	1.1	9.4	4.2	1.1
5.0	44.5	25.5	12.6	45.6	26.9	14.0	48.6	29.8	16.9
	18.7	7.8	2.4	13.6	12.6	18.5	34.1	49.5	80.1
	23.2	14.4	7.9	23.2	14.4	7.9	23.2	14.4	7.9

1) low-to-high mode
2) high-to-low mode
3) constant-stress test

5. Concluding Remarks

This paper presented optimal simple low-to-high step and constant-stress ALT plans for the items having lognormally-distributed lifetimes. Low stress and stress change times or sample proportion allocated at low stress minimizing the asymptotic variance of MLE of the acceleration factor are determined. Our results show that, one can be free to choose between the two modes of step-stress tests and constant-stress test, it may be natural to compare the corresponding saymptotic variances and PAVIs. Designing the step and constant-stress ALTs for items whose lifetime's scale parameter is not constant but varying with stress is under study.

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Appendix

1. *Fisher information matrix for a low-to-high step-stress test*

Let $\lambda'(\varphi) \equiv \partial\lambda(\varphi)/\partial\varphi = \lambda^2(\varphi) - \varphi\phi(\varphi)\lambda(\varphi)$, $\sigma z_1'(y_i) \equiv \sigma(\partial z_1(y_i)/\partial\gamma_1) = -s_1$, $\sigma z_2'(y_i) \equiv \sigma(\partial z_2(y_i)/\partial\gamma_1) = -z_3(y_i) = (1-s_1)\delta_L/(y_i-\tau_L+\delta_L)-1$, $z_4'(y_i) \equiv \partial z_4(y_i)/\partial\gamma_1 = -\partial z_3(y_i)/\partial\gamma_1 = (1-s_1)^2\delta_L/(y_i-\tau_L+\delta_L) - (1-s_1)^2\delta_L^2/(y_i-\tau_L+\delta_L)^2$ and $\omega \equiv -\partial\rho/\partial\gamma_1 = (1-s_1)^2(\eta-\tau_L)\delta_L/(\eta-\tau_L+\delta_L)^2$. Then the second partial derivatives of l_i with respect to γ_0, γ_1 and σ are as follows:

$$-\sigma^2(\partial^2 l_i/\partial\gamma_0^2) = \sum_{j=1}^{2} I_{ij} + I_{i3}\lambda'(\varphi) \quad \text{(A-1)}$$

$$-\sigma^2(\partial^2 l_i/\partial\gamma_0\partial\gamma_1) = I_{i1}s_1 + I_{i2}\{1-(1-s_1)\delta_L/(y_i-\tau_L+\delta_L)\} + I_{i3}\rho\lambda(\varphi) \quad \text{(A-2)}$$

$$-\sigma^2(\partial^2 l_i/\partial\gamma_0\partial\sigma) = 2\sigma(\partial l_i/\partial\gamma_0) + I_{i3}\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\} \quad \text{(A-3)}$$

$$-\sigma^2(\partial^2 l_i/\partial\gamma_1^2) = s_1^2 I_{i1} + I_{i2}[\{\sigma^2 + (\ln(y_i - \tau_L + \delta_L) - \gamma_0 - \gamma_1)\}\cdot \{(1-s_1)^2\delta_L/(y_i - \tau_L + \delta_L) - (1-s_1)^2\delta_L^2/(y_i - \tau_L + \delta_L)^2\} + \{(1-s_1)^2\delta_L^2/(y_i - \tau_L + \delta_L)^2 - 2(1-s_1)\delta_L/(y_i - \tau_L + \delta_L) + 1\}] + I_{i3}\{\lambda(\varphi)\rho^2 + \sigma\omega\lambda(\varphi)\} \quad \text{(A-4)}$$

$$-\sigma^2(\partial^2 l_i/\partial\gamma_1\partial\sigma) = 2\sigma(\partial l_i/\partial\gamma_1) + I_{i2}2\sigma(1-s_1)\delta_L/(y_i - \tau_L + \delta_1) + I_{i3}\rho\{\varphi\lambda'(\varphi) - \lambda(\varphi)\} \quad \text{(A-5)}$$

$$-\sigma^2(\partial^2 l_i/\partial\sigma^2) = 3\sigma(\partial l_i/\partial\sigma) + 2\sum_{j=1}^{2} I_{ij} + I_{i3}\varphi\{-\lambda(\varphi) + \varphi\lambda'(\varphi)\} \quad \text{(A-6)}$$

The Fisher information matrix can be obtained by taking expectations of the equations (A-1)~(A-6). The expectations required to obtain the Fisher information matrix are as follows:

$$E(I_{i1}) = \int_0^{\tau_1} f(y)dy = \Phi(\zeta) = p_1 \quad \text{(A-7)}$$

$$E(I_{i2}) = \int_{\tau_1}^{\eta} f(y)dy = \Phi(\varphi) - \Phi(\zeta) = p_2 \quad \text{(A-8)}$$

$$E(I_{i3}) = E(1 - I_{i1} - I_{i2}) = 1 - \Phi(\varphi) = p_s \quad \text{(A-9)}$$

$$E\{I_{i2}(y_i - \tau_L + \delta_L)^{-j}\} = \int_\zeta^\varphi e^{-j(\gamma_0 + \gamma_1 + \sigma z)}\phi(z)dz = e^{-j(\gamma_0 + \gamma_1) + \frac{(j\sigma)^2}{2}}\{\Phi(\varphi + j\sigma) - \Phi(\zeta + j\sigma)\},\ j=1,2 \quad \text{(A-10)}$$

$$E[I_{i2}\{(\ln(y_i - \tau_L + \delta_L) - \gamma_0 - \gamma_1)/\sigma\}(y_i - \tau_L + \delta_L)^{-j}] = \int_\zeta^\varphi e^{-j(\gamma_0 + \gamma_1 + \sigma z)} z\phi(z)dz = e^{-j(\gamma_0 + \gamma_1) + \frac{(j\sigma)^2}{2}}[\phi(\zeta + j\sigma) - \phi(\varphi + j\sigma) - j\sigma\{\Phi(\varphi + j\sigma) - \Phi(\zeta + j\sigma)\}],\ j = 1,2 \quad \text{(A-11)}$$

2. *Fisher information matrix for a high-to-low step-stress test*

The lifetime distribution of Y, the random variable representing the total lifetime of an item under a simple high-to-low mode test, is

$$f(y)=\begin{cases} 0, & y \le 0 \\ \phi(z_1(y)) \ / \ (\sigma y), & 0 < y \le \tau_H, \\ \phi(z_2(y)) \ / \ (\sigma(y-\tau_H+\delta_H)), & \tau_H < y \end{cases} \qquad \text{(A-12)}$$

where $z_1(y) = (\ln y-\gamma_0-\gamma_1) \ / \ \sigma$, and $z_2(y) = (\ln(y-\tau_H+\delta_H)-\gamma_0-\gamma_1 s_1))/\sigma$. By using the p.d.f. of (A-12) the Fisher information matrix can be obatained in a similar way as in the low-to-high mode test.

3. *Fisher information matrix in terms of the design parameters*

Each element of the Fisher information matrix for a low-to-high step-stress test can be expressed in terms of the following four quantities, which are functions of the design parameters of s_1, x_L, and (p_d, p_h, σ).

1) $\zeta = (\ln(\tau_L/\eta)+\ln\eta-\gamma_0-\gamma_1 s_1)/\sigma = \dfrac{\ln x_L}{\sigma} + p_d + (p_h - p_d)s_1$

2) $\varphi = (\ln((\eta-\tau_L+\delta_L)/\eta)+\ln\eta-\gamma_0-\gamma_1)/\sigma = \dfrac{\ln(1-x_L+\delta_L^0)}{\sigma} + p_h$

where $\delta_L^0 = \delta_L/\eta = x_L e^{-(1-s_1)\sigma(p_h-p_d)}$

3) $\rho = \dfrac{(\eta-\tau_L+s_1\delta_L)/\eta}{(\eta-\tau_L+\delta_L)/\eta} = \dfrac{1-x_L+s_1\delta_L^0}{1-x_L+\delta_L^0}$

4) $\omega = \dfrac{(1-s_1)^2(\eta-\tau_L)\delta_L/\eta^2}{(\eta-\tau_L+\delta_L)^2/\eta^2} = \dfrac{(1-s_1)^2(1-x_L)\delta_L^0}{(1-x_L+\delta_L^0)^2}$

The Fisher information matrix for a high-to-low step-stress testcan be expressed in terms of the following four quantities, which are functions of the design parameters of s_1, x_H, and (p_d, p_h, σ).

5) $\zeta = (\ln(\tau_H/\eta)+\ln\eta-\gamma_0-\gamma_1)/\sigma = \dfrac{\ln x_H}{\sigma} + p_h$

6) $\varphi = (\ln((\eta - \tau_H + \delta_H)/\eta) + \ln\eta - \gamma_0 - \gamma_1 s_1)/\sigma$

$$= \frac{\ln\ (1 - x_H + \delta_{H0})}{\sigma} + p_d + (p_h - p_d)s_1$$

where $\delta_H^0 = \delta_H/\eta = (\tau_H/\eta)e^{\gamma_1(s_1-1)} = x_H e^{-(s_1-1)\sigma(p_h-p_d)}$

7) $\rho = \dfrac{s_1(\eta - \tau_H)/\eta + \delta_H/\eta}{(\eta - \tau_H + \delta_H)/\eta} = \dfrac{s_1(1-x_H) + \delta_H^0}{1 - x_H + \delta_H^0}$

8) $\omega = \dfrac{(s_1-1)^2(\eta - \tau_H)\delta_H/\eta^2}{(\eta - \tau_H + \delta_H)^2/\eta^2} = \dfrac{(s_1-1)^2(1-x_H)\delta_H^0}{(1 - x_H + \delta_H^0)^2}$