⊠연구논문

Periodic Preventive Maintenance Policies when Minimal Repair Costs Vary at Failures

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Abstract

This paper considers a repairable system, which is maintained preventively at periodic times and is minimally repaired at each failure. Most preventive maintenance policies for such repairable systems assume that the cost of minimal repair is constant regardless of its age at failure. However, it is more practical to consider the situations where the cost of minimal repair is dependent not only on its age at failure, but also on the number of preventive maintenance carried out prior to its failure. We consider the preventive maintenance policy with age-dependent minimal repair cost. The optimal policies which minimize the expected cost rate over an infinite time span are discussed. We obtain the optimal period and number of preventive maintenance prior to replacement of the system.

1. Introduction

Preventive maintenance(PM) of a repairable system is of great interest among engineers and reliability analysts. For a complex system, the maintenance action is not necessarily the replacement of the whole system, but is often repaired to the functioning state at each failure. Hence the system may not be restored to as good as new immediately after the completion of maintenance action.

PM policies have been discussed extensively in the literature by many authors. Barlow and Hunter(1960) consider a PM policy of periodic replacement with minimal repair at any intervening failures. Nguyen and Murthy(1981) study two types of PM policies for a repairable system and assume that the life distribution of a system changes after each repair in such a way that its failure rate increases with the number of repairs carried out. They obtain the optimal policies to minimize the expected cost per unit time for an infinite time span. Murthy and Nguyen(1981) study the optimal age replacement policy with imperfect preventive maintenance. The preventive is imperfect in the sense that it can cause failure of a non-failed system. Berg(1984) proposes a preventive replacement policy, called a modified age replacement policy.

Nakagawa(1986) considers periodic and sequential preventive maintenance policies for the system with minimal repair at failure: the PM is done (i) at periodic times kx and (ii) at constant intervals $x_k (k = 1, 2, \cdots, N)$. The system is replaced by a new system at the Nth PM. If the system fails between PM, it undergoes minimal repair and thus its failure rate is not affected by such minimal repairs. Nakagawa(1986), however, assumes that the failure rate increase with the number of PM. It is also assumed that the cost of minimal repair at intervening failure is constant, independent of its age at failure. Most PM for a repairable system with minimal repair at failure assume that the cost of minimal repair is constant regardless of its age at failure. However, such assumption might not hold true for real situations. It is more reasonable to assume that the minimal repair cost is dependent not only on its age at failure, but also on the number of PM undergone prior to its failure.

In this paper, we consider a preventive maintenance policy for a repairable system when minimal repair cost varies with time. The system is maintained preventively at periodic times $kx (k = 1, 2, \cdots, N)$ and is replaced at the Nth PM. Between PM, the system is assumed to have different intervals and has a different failure distribution between PM. If the system fails between PM, it undergoes only minimal repair and hence, the failure rate remains undisturbed by any of these minimal repairs. We assume that the cost of minimal repair is a function of system age and the number of PM. The expression to compute the expected cost rate per unit time is given. We also obtain the optimal period x and number N for the periodic PM, which minimize the expected cost rate per unit time for an infinite time span.

In Section 2, we present the expression for the expected cost rate for the periodic PM and consider several types of cost functions. Section 3 discusses the optimal periodic PM policies to obtain the optimal x and N. Section 4 gives the explicit solutions for the optimal periodic PM policies when the failure time follows a Weibull distribution.

2. Expected Cost Rate

Suppose that the cost of minimal repair at the intervening failures between PM depends on its age t at failure and the number of PM, k, undergone prior to its failure. Let $C_k(t)$ $(k=1,2,\cdots,N)$ denote the cost of minimal repair in the kth period of PM for $t \geq 0$, where t denotes the time elapsed after the $(k-1)$th PM and $C_k(t)$ is assumed to be a continuous function of t. Let F denote the life distribution function with its density f and let $r_k(t)$ denote the failure rate function in the kth period of PM. The hazard function in the kth period of PM is defined as $R_k(t) = \int_0^t r_k(s)\, ds$.

To obtain the optimal period x and number N for the periodic PM we assume that for $k=1,2,\cdots,N$, $C_k(t)\, r_k(t) < C_{k+1}(t)\, r_{k+1}(t)$ for any $t>0$, i.e., the product of cost and failure rate increases with the number of PM. Assuming that the necessary times to conduct PM, minimal repair and replacement are negligible, the expected cost rate for running our periodic PM policy can be obtained in the following two useful expressions.

$$C(x,N) = \frac{\sum_{k=1}^{N} \int_0^x C_k(t)\, r_k(t)\, dt + (N-1)c_2 + c_3}{Nx} = \frac{\sum_{k=1}^{N} \int_0^{R_k(x)} C_k(R_k^{-1}(t))\, dt + (N-1)c_2 + c_3}{Nx}, \tag{2.1}$$

where $C_k(t)$ is the cost of minimal repair in the kth period of PM, c_2 is the cost of PM, and c_3 is the cost of replacement with $c_3 \geq c_2$.

The formula (2.1) can be obtained by combining the results given in Boland (1982) and Nakagawa(1986). The second expression of (2.1) is easily obtained by using the change of variable. $\int_0^x C_k(t)\, r_k(t)\, dt$ can be interpreted as the expected cost on minimal repair during $[0, x]$ in the kth period of PM. For more detailed discussions of such integration, see Boland(1982).

In Examples 1 and 2, we consider two different types of minimal repair cost functions and determine the resulting expected cost rates, using the formula (2.1).

Example 1. The cost function is of the type $C_k(t) = ct^{\alpha}$, $\alpha \geq 0$. Then the expected cost rates is obtained as

$$C(x, N) = \frac{\sum_{k=1}^{N} \int_0^x ct^{\alpha} r_k(t)\, dt + (N-1)c_2 + c_3}{Nx}$$
$$= \frac{\sum_{k=1}^{N} \left\{ cx^{\alpha} R_k(x) - \int_0^x c\alpha t^{\alpha-1} R_k(t)\, dt \right\} + (N-1)c_2 + c_3}{Nx}.$$

In the particular case when $\alpha = 1$, the expected cost rate is

$$C(x, N) = \frac{\sum_{k=1}^{N} \left\{ cxR_k(x) - \int_0^x cR_k(t)\, dt \right\} + (N-1)c_2 + c_3}{Nx}.$$

Example 2. The cost function is of the type $C_k(t) = g(R_k(t))$, where $R_k(t)$ is the hazard function in the kth period of PM. We consider the following particular cases.

1) $g(y) = cy^{\alpha}$, $\alpha \geq 0$.
Then the expected cost rate is

$$C(x, N) = \frac{\sum_{k=1}^{N} \left\{ \frac{cR_k^{\alpha+1}(x)}{\alpha+1} \right\} + (N-1)c_2 + c_3}{Nx}.$$

When $\alpha = 1$, i.e., the minimal repair cost is proportional to the hazard function, the expected cost rate is

$$C(x, N) = \frac{\sum_{k=1}^{N} \left\{ \frac{cR_k^2(x)}{2} \right\} + (N-1)c_2 + c_3}{Nx}.$$

When $\alpha = 0$, that is, the cost function is constant, the expected cost rate becomes

$$C(x, N) = \frac{\sum_{k=1}^{N} cR_k(x) + (N-1)c_2 + c_3}{Nx},$$

which is the expected cost rate considered in Nakagawa(1986).

2) $g(y) = ce^{\alpha y},\ \alpha \ge 0.$

The expected cost rate is

$$C(x,N) = \frac{\sum_{k=1}^{N}\left\{\frac{c}{\alpha}\left(e^{\alpha R_k(x)} - 1\right)\right\} + (N-1)c_2 + c_3}{Nx}$$

$$= \frac{\sum_{k=1}^{N}\left\{\frac{c}{\alpha}\left(\frac{1}{S_k^{\alpha}(x)} - 1\right)\right\} + (N-1)c_2 + c_3}{Nx},$$

where $S_k(t) = 1 - F_k(t)$.

When $\alpha = 1$, then $C_k(t) = ce^{R_k(t)} = c \cdot 1 / S_k(t)$, i.e., the minimal repair cost is inversely proportional to the survival probability. In this case the expected cost rate becomes

$$C(x,N) = \frac{\sum_{k=1}^{N}\left\{c\left(\frac{1}{S_k(x)} - 1\right)\right\} + (N-1)c_2 + c_3}{Nx}$$

$$= \frac{\sum_{k=1}^{N}\left(c\frac{F_k(x)}{S_k(x)}\right) + (N-1)c_2 + c_3}{Nx}.$$

Note that $F_k(x) / S_k(x)$ is the odds ratio in the kth period of PM.

3) $g(y) = c(y^{\alpha} + e^{\beta y}),\ \alpha \ge 0,\ \beta \ge 0$, a linear combination of 1) and 2).

For this case the minimal repair cost is of the type

$$C_k(t) = g(R_k(t)) = c\left(R_k^{\alpha}(t) + e^{\beta R_k(t)}\right).$$

Thus, the expected cost rate is

$$C(x,N) = \frac{\sum_{k=1}^{N}\left\{c\frac{R_k^{\alpha+1}(x)}{\alpha+1} + c\frac{e^{\beta R_k(x)}}{\beta} - \frac{c}{\beta}\right\} + (N-1)c_2 + c_3}{Nx}$$

$$= \frac{\sum_{k=1}^{N}\left[c\left\{\frac{R_k^{\alpha+1}(x)}{\alpha+1} + \frac{1}{\beta}\left(e^{\beta R_k(x)} - 1\right)\right\}\right] + (N-1)c_2 + c_3}{Nx}.$$

If $\alpha = 0$ and $\beta = 1$, then the expected cost rate results in

$$C(x, N) = \frac{\sum_{k=1}^{N} c\{ R_k(x) + e^{R_k(x)} - 1\} + (N-1)c_2 + c_3}{Nx} .$$

In Section 3, we prove the existence and uniqueness of x and N which minimize the expected cost rate over an infinite time span under some conditions on $C_k(t) r_k(t)$.

3. Optimal Periodic PM Policy

We now consider the problem of finding the optimal period x^* and number of PM, N^*, prior to the replacement of the system which minimize $C(x, N)$, given in (2.1). To find a N^* which minimizes $C(x, N)$ for given $x \geq 0$, we form the inequalities $C(x, N+1) \geq C(x, N)$ and $C(x, N) < C(x, N-1)$. These inequalities imply

$$N \int_0^x C_{N+1}(t) r_{N+1}(t) dt - \sum_{k=1}^{N} \int_0^x C_k(t) r_k(t) dt \geq c_3 - c_2$$

and

$$(N-1) \int_0^x C_N(t) r_N(t) dt - \sum_{k=1}^{N-1} \int_0^x C_k(t) r_k(t) dt < c_3 - c_2 .$$

Thus, we have

$$L(x, N) \geq (c_3 - c_2) \quad \text{and} \quad L(x, N-1) < (c_3 - c_2), \tag{3.1}$$

where

$$L(x, N) = \begin{cases} N \int_0^x C_{N+1}(t) r_{N+1}(t) dt - \sum_{k=1}^{N} \int_0^x C_k(t) r_k(t) dt , & N = 1, 2, \cdots \\ 0 & N = 0 . \end{cases}$$

Theorem 1. Suppose that $C_k(t)\,r_k(t)$ is strictly increasing in k and tends to ∞ as $k\to\infty$ for fixed $t\geq 0$. Then there exists a finite and unique N^* which satisfies (3.1) for any $x>0$.

Proof. From the assumption that $C_k(t)\,r_k(t) < C_{k+1}(t)\,r_{k+1}(t)$ for $t\geq 0$ and the relations (3.1), we obtain

$$\begin{aligned} L(x,N)-L(x,N-1) &= N\int_0^x [\,C_{N+1}(t)\,r_{N+1}(t) - C_N(t)\,r_N(t)\,]\,dt \\ &> 0 \end{aligned}$$

and

$$L(x,N) \geq \int_0^x C_{N+1}(t)\,r_{N+1}(t)\,dt - \int_0^x C_1(t)\,r_1(t)\,dt.$$

Thus, $L(x,N)$ is strictly increasing in N and tends to ∞ as $N\to\infty$, and hence the result follows.

It immediately follows from Theorem 1 that

Corollary 1. Let $t>0$ be given. If $C_k(t)$ and $r_k(t)$ are strictly increasing in k and $\lim_{k\to\infty} r_k(t)=\infty$, then there exists a finite and unique N^* which satisfies (3.1) for any $x>0$.

Next, we find the optimal period x^*. Differentiating $C(x,N)$ with respect to x and setting it equal to 0, we have

$$\sum_{k=1}^{N}\left[x C_k(x)\,r_k(x) - \int_0^x C_k(t)\,r_k(t)\,dt\right] = (N-1)\,c_2 + c_3. \tag{3.2}$$

Our objective is to determine if there exists a value of x which satisfies (3.2). If such an x exists, it is the optimal period for our periodic PM policy.

Theorem 2. Assume that $C_k(t)$ and $r_k(t)$ are differentiable with respect to t and $C_k(t)\,r_k(t)$ is strictly increasing to ∞ as t increases to ∞. Then there exists a finite and unique x^* which satisfies (3.2) for any integer N.

Proof. The left-hand side of (3.2) is strictly increasing to ∞, since

$$\begin{aligned}\left\{\sum_{k=1}^{N}\left[x\ C_k(x)\, r_k(x) - \int_0^x C_k(t)\, r_k(t)\, dt\right]\right\}' \\ = x\sum_{k=1}^{N}[\, C_k{}'(x)\, r_k(x) + C_k(x)\, r_k{}'(x)\,] \\ = x\sum_{k=1}^{N}\{C_k(x)\, r_k(x)\,\}' > 0\end{aligned} \tag{3.3}$$

and for $x_1 < x$,

$$\begin{aligned}\sum_{k=1}^{N}\left[x C_k(x)\, r_k(x) - \int_0^x C_k(t)\, r_k(t)\, dt\right] \\ > \sum_{k=1}^{N}\left[x_1 C_k(x)\, r_k(x) - \int_0^{x_1} C_k(t)\, r_k(t)\, dt\right].\end{aligned} \tag{3.4}$$

The relation (3.4) implies that the left-hand side of (3.2) becomes ∞ as $x \to \infty$. Thus, there exists a finite and unique x^* which satisfies (3.2) for any integer N.

Note that if $C_k(t)$ and $r_k(t)$ are strictly increasing in $t \geq 0$ and $\lim_{t\to\infty} r_k(t) = \infty$ for a given $k \geq 1$, then there exists a finite and unique x^* which satisfies (3.2) for any integer N.

4. Numerical Example

For the purpose of comparison of our optimal periodic PM policy with that of Nakagawa(1986), we use the same life distribution as in Nakagawa(1986).

Suppose that the failure time of the system has a Weibull distribution, i.e., $r_k(t) = \alpha_k \beta t^{\beta-1}$ for $\beta > 1$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_{N+1}$. As a special case, we take $\beta = 2$, $\alpha_k = 1/\ 100 \times (0.81)^{k-1}$ and $C_k(t) = g(R_k(t))$ for $k = 1, 2, \cdots$, where $g(y) = cy$ and $c > 0$ is a constant. This type of cost function is discussed in case 1 of Example 2. Since $R_k(t)$ is equal to $a_k t^2$, we have $C_k(t) = ca_k t^2$. Then $C_k(t) r_k(t)$ is increasing in k and also is strictly increasing to ∞ as $t \to \infty$. Thus, there exists a unique x^* and N^* of periodic PM which minimizes the expected cost rate per unit time over an infinite time span.

It can easily be shown that (3.1) and (3.2) imply

$$N(a_{N+1})^2 \frac{x^4}{2} - \sum_{k=1}^{N} \frac{(a_k)^2}{2} x^4 \geq \frac{c_3 - c_2}{c} . \tag{4.1}$$

and

$$x^4 = \frac{(N-1)\,c_2 + c_3}{\left(\frac{3}{2}c\right) \sum_{k=1}^{N} (a_k)^2} . \tag{4.2}$$

From (4.1) and (4.2), N^* can be determined as the smallest N satisfying

$$\left(\frac{(N-1)\,c_2 + c_3}{3}\right)\left(\frac{N(a_{N+1})^2}{\sum_{k=1}^{N} (a_k)^2} - 1\right) \geq c_3 - c_2 . \tag{4.3}$$

Once the value of N^* is determined, the optimal period x^* is obtained as

$$x^* = \left(\frac{(N^* - 1)\,c_2 + c_3}{\left(\frac{3}{2}c\right) \sum_{k=1}^{N^*} (a_k)^2}\right)^{\frac{1}{4}} . \tag{4.4}$$

Replacing x and N of (2.1) by x^* and N^*, we obtain the corresponding expected cost rate as follows.

$$C(x^*, N^*) = \frac{2c}{N^*} (x^*)^3 \left(\sum_{k=1}^{N^*} (a_k)^2\right). \tag{4.5}$$

<Table 1> presents the optimal x^* and N^* of periodic PM for various values of c_3 when $c_2 = 3$ and $c = 1$. The corresponding expected cost rate is also provided.

< Table 1 > Optimal time x^*, number N^* and its expected cost rate $C(x^*, N^*)$ with $c_2 = 3$ and $c = 1$.

	c_3											
	3	4	5	6	7	8	9	10	11	12	13	14
N^*	1	2	2	3	3	3	4	4	4	4	4	4
x^*	11.99	11.66	12.06	11.33	11.56	11.78	10.94	11.09	11.23	11.37	11.50	11.63
$C(x^*, N^*)$	0.336	0.440	0.443	0.470	0.499	0.528	0.549	0.572	0.594	0.616	0.638	0.660

	c_3											
	15	16	17	18	19	20	30	32	40	50	75	77
N^*	4	4	5	5	5	5	5	5	6	6	6	6
x^*	11.75	11.87	10.88	10.98	11.07	11.15	11.94	12.08	11.36	11.84	12.85	12.92
$C(x^*, N^*)$	0.680	0.701	0.710	0.730	0.748	0.764	0.939	0.972	1.076	1.218	1.557	1.583

It is interesting to note that as the cost of replacement becomes higher, the required number of PM prior to replacement gets larger and the period x^* and $C(x^*, N^*)$ change very little and remain almost constant. However, if the number of periodic PM is fixed then the period x^* and $C(x^*, N^*)$ get larger as the cost of replacement becomes higher.

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