Fuzzy Pairwise Almost Continuous Mappings on Fuzzy Bitopological Spaces

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Ⅰ. Introduction

The concept of a fuzzy set was introduced by Zadeh in his classical paper [12]. Using the concept of a fuzzy set, Chang [4] introduced a fuzzy topological space. Since then, many mathematicians have contributed to the development of this theory. Azad [3] introduced a fuzzy semiopen set and a fuzzy regular open set, and he investigated a fuzzy semicontinuous mapping and a fuzzy almost continuous mapping on fuzzy topological spaces.

Kandil [5] introduced and studied a fuzzy bitopological space as a natural generalization of a fuzzy topological space. Sampath Kumar [9] introduced a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy semiopen set and a fuzzy pairwise semicontinuous mapping on fuzzy bitopological spaces, and investigated some of their basic properties.

In this paper, after we define a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy regular open set, we study a fuzzy pairwise almost continuous mapping on fuzzy bitopological spaces. It can be shown that every $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy regular open (respectively $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy regular closed) set is a $\mathcal{T}_i$-fuzzy open (respectively $\mathcal{T}_i$-fuzzy closed) set, but the converse is not true in general. The intersection (respectively union) of two $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy regular open (respectively $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy regular closed) sets is a $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy regular open (respectively $(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy regular closed) set. We show that every fuzzy pairwise continuous mapping is a fuzzy pairwise almost continuous mapping, but the converse is not true in general.

Ⅱ. Preliminaries

A system $(X, \mathcal{T}_1, \mathcal{T}_2)$ consisting of a set X with two fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X is called a **fuzzy bitopological space** [*fbts*] X [5]. Throughout this paper, indices i, j take values in $\{1, 2\}$ with $i \neq j$.

Let $f:(X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{T}_1^*, \mathcal{T}_2^*)$ be a mapping. f is called a **fuzzy pairwise continuous** [*fpc*] **mapping** if the induced mappings $f:(X, \mathcal{T}_k) \to (Y, \mathcal{T}_k^*)$ $(k=1, 2)$ are fuzzy continuous mappings [9].

Definition 2.1. [9] Let μ be a fuzzy set of a *fbts* X. Then μ is called;

(i) a **$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy semiopen** [$(\mathcal{T}_i, \mathcal{T}_j)-fso$] **set** of X if there exists a $\mathcal{T}_i$-fuzzy open [$\mathcal{T}_i-fo$] set ν of X such that $\nu \leq \mu \leq \mathcal{T}_j-\mathrm{Cl}\,\nu$,

(ii) a **$(\mathcal{T}_i, \mathcal{T}_j)$-fuzzy semiclosed** [$(\mathcal{T}_i, \mathcal{T}_j)-fsc$] **set** of X if there exists a $\mathcal{T}_i$-fuzzy closed [$\mathcal{T}_i-fc$] set ν of X such that $\mathcal{T}_j-\mathrm{Int}\,\nu \leq \mu \leq \nu$.

Lemma 2.2. [9] Let μ be a fuzzy set of a *fbts* X. Then the following statements are equivalent:

(i) μ is a $(\mathcal{T}_i, \mathcal{T}_j)-fsc$ set.

(ii) μ^c is a $(\mathcal{T}_i, \mathcal{T}_j)-fso$ set.

(iii) $\mathcal{T}_j-\mathrm{Int}(\mathcal{T}_i-\mathrm{Cl}\mu) \leq \mu$.

(iv) $\mathcal{T}_j-\mathrm{Cl}(\mathcal{T}_i-\mathrm{Int}\,\mu^c) \geq \mu^c$.

Proposition 2.3. [9] (i) Any union of $(\mathcal{T}_i, \mathcal{T}_j)-fso$ sets is a $(\mathcal{T}_i, \mathcal{T}_j)-fso$ set.

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(ii) Any intersection of $(\mathcal{I}_i, \mathcal{I}_j)-fsc$ sets is a $(\mathcal{I}_i, \mathcal{I}_j)-fsc$ set.

It is clear that every $\mathcal{I}_i-fo$ (respectively $\mathcal{I}_i-fc$) set is a $(\mathcal{I}_i, \mathcal{I}_j)-fso$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-fsc$) set, but the converse need not be true [9]. The intersection (respectively union) of any two $(\mathcal{I}_i, \mathcal{I}_j)-fso$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-fsc$) sets needs not be a $(\mathcal{I}_i, \mathcal{I}_j)-fso$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-fsc$) set [9]. Even the intersection (respectively union) of a $(\mathcal{I}_i, \mathcal{I}_j)-fso$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-fsc$) set with a $\mathcal{I}_i-fo$ (respectively $\mathcal{I}_i-fc$) set may fail to be a $(\mathcal{I}_i, \mathcal{I}_j)-fso$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-fsc$) set [9].

Definition 2.4. [9] Let $f:(X, \mathcal{I}_1, \mathcal{I}_2)\to(Y, \mathcal{I}_1^*, \mathcal{I}_2^*)$ be a mapping. Then f is called a **fuzzy pairwise semi-continuous** [*fpsc*] **mapping** if $f^{-1}(\nu)$ is a $(\mathcal{I}_i, \mathcal{I}_j)-fso$ set of X for each $\mathcal{I}_i^*-fo$ set ν of Y.

From the above definition we know that every *fpc* mapping is a *fpsc* mapping, but the converse is not true in general [9].

III. Fuzzy pairwise almost continuous mappings

We define a $(\mathcal{I}_i, \mathcal{I}_j)$-fuzzy regular open set, introduce a fuzzy pairwise almost continuous mapping on fuzzy bitopological spaces and investigate some of their properties.

Definition 3.1. Let μ be any fuzzy set of a *fbts* X. Then μ is called;

(i) a **$(\mathcal{I}_i, \mathcal{I}_j)$-fuzzy regular open** $[(\mathcal{I}_i, \mathcal{I}_j)-fro]$ **set** of X if $\mu=\mathcal{I}_i-\mathrm{Int}(\mathcal{I}_j-\mathrm{Cl}\,\mu)$,

(ii) a **$(\mathcal{I}_i, \mathcal{I}_j)$-fuzzy regular closed** $[(\mathcal{I}_i, \mathcal{I}_j)-frc]$ **set** of X if $\mu=\mathcal{I}_i-\mathrm{Cl}(\mathcal{I}_j-\mathrm{Int}\,\mu)$.

It is clear that every $(\mathcal{I}_i, \mathcal{I}_j)-fro$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-frc$) set is a $\mathcal{I}_i-fo$ (respectively $\mathcal{I}_i-fc$) set. But the converse need not be true by the following example. This example also shows that the union (respectively intersection) of two $(\mathcal{I}_i, \mathcal{I}_j)-fro$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-frc$) sets need not be a $(\mathcal{I}_i, \mathcal{I}_j)-fro$ (respectively $(\mathcal{I}_i, \mathcal{I}_j)-frc$) set.

Example 3.2. Let μ_1, μ_2, μ_3, μ_4, μ_5, μ_6 and μ_7 be fuzzy sets of $X=\{a, b, c\}$, defined as follows;

$\mu_1(a)=0.7,\ \mu_1(b)=0.7,\ \mu_1(c)=0.2,$
$\mu_2(a)=0.6,\ \mu_2(b)=0.7,\ \mu_2(c)=0.7,$
$\mu_3(a)=0.0,\ \mu_3(b)=0.0,\ \mu_3(c)=0.7,$
$\mu_4(a)=0.4,\ \mu_4(b)=0.3,\ \mu_4(c)=0.0,$
$\mu_5(a)=0.4,\ \mu_5(b)=0.3,\ \mu_4(c)=0.2,$
$\mu_6(a)=0.8,\ \mu_6(b)=0.8,\ \mu_6(c)=0.8,$
$\mu_7(a)=0.9,\ \mu_7(b)=0.9,\ \mu_7(c)=0.8.$

Let $\mathcal{I}_1=\{0_X, \mu_1, \mu_2, \mu_1\vee\mu_2, \mu_1\wedge\mu_2, \mu_3, \mu_1\wedge\mu_3, \mu_4, \mu_3\vee\mu_4, \mu_5, \mu_6, 1_X\}$ and $\mathcal{I}_2=\{0_X, \mu_1, \mu_2, \mu_1\vee\mu_2, \mu_1\wedge\mu_2, \mu_3, \mu_1\wedge\mu_3, \mu_4, \mu_3\vee\mu_4, \mu_5, \mu_7, 1_X\}$ be fuzzy topologies on X. Then μ_1 and μ_2 are $(\mathcal{I}_i, \mathcal{I}_j)-fro$ sets but $\mu_1\vee\mu_2$ is not a $(\mathcal{I}_i, \mathcal{I}_j)-fro$ set. Also, $(\mu_1\vee\mu_2)^c=\mu_1^c\vee\mu_2^c$ is not a $(\mathcal{I}_i, \mathcal{I}_j)-frc$ set of *fbts* X. □

Proposition 3.3. (i) The intersection of two $(\mathcal{I}_i, \mathcal{I}_j)-fro$ sets is a $(\mathcal{I}_i, \mathcal{I}_j)-fro$ set.

(ii) The union of two $(\mathcal{I}_i, \mathcal{I}_j)-frc$ sets is a $(\mathcal{I}_i, \mathcal{I}_j)-frc$ set.

Proof. (i) Let μ and ν be any $(\mathcal{I}_i, \mathcal{I}_j)-fro$ sets of a *fbts* X. Since $\mu\wedge\nu$ is a $\mathcal{I}_i-fo$ set, $\mu\wedge\nu\le\mathcal{I}_i-\mathrm{Int}(\mathcal{I}_j-\mathrm{Cl}(\mu\wedge\nu))$. Now,

$$\mathcal{I}_i-\mathrm{Int}(\mathcal{I}_j-\mathrm{Cl}(\mu\wedge\nu))\le\mathcal{I}_i-\mathrm{Int}(\mathcal{I}_j-\mathrm{Cl}\,\mu)=\mu$$

and

$$\mathcal{I}_i-\mathrm{Int}(\mathcal{I}_j-\mathrm{Cl}(\mu\wedge\nu))\le\mathcal{I}_i-\mathrm{Int}(\mathcal{I}_j-\mathrm{Cl}\,\nu)=\nu.$$

Hence $\mathcal{I}_i-\mathrm{Int}(\mathcal{I}_j-\mathrm{Cl}(\mu\wedge\nu))\le\mu\wedge\nu$. Therefore $\mu\wedge\nu$ is a $(\mathcal{I}_i, \mathcal{I}_j)-fro$ set.

(ii) It follows easily by taking complements of μ and ν in (i). □

Definition 3.4. Let $f:(X, \mathcal{I}_1, \mathcal{I}_2)\to(Y, \mathcal{I}_1^*, \mathcal{I}_2^*)$ be a mapping. Then f is called a **fuzzy pairwise almost continuous** [*fpac*] **mapping** if $f^{-1}(\nu)$ is a $\mathcal{I}_i-fo$ set of X for each $(\mathcal{I}_i^*, \mathcal{I}_j^*)-fro$ set ν of Y.

Clearly every *fpc* mapping is a *fpac* mapping. But the converse need not be true by the following example. Also, Example 3.5 and Example 3.6 show that a *fpsc* mapping and a *fpac* mapping do not have any specific relations.

Example 3.5. Let λ, μ and ν be fuzzy sets of I, defined as follows; for each $x\in I$,

$$\lambda(x)=x,\ \mu(x)=1-x \text{ and } \nu(x)=\begin{cases} x, & 0\le x\le \frac{1}{2} \\ 0, & \frac{1}{2}<x\le 1. \end{cases}$$

Consider fuzzy topologies $\mathcal{I}_1=\{0_X, \lambda, \mu, \lambda\vee\mu, \lambda\wedge\mu, 1_X\}$, $\mathcal{I}_2=\{0_X, \nu, \lambda\vee\mu, \lambda\wedge\mu, 1_X\}$, $\mathcal{I}_1^*=\{0_X, \lambda, \mu, \nu, \lambda\vee\mu, \lambda\wedge\mu, 1_X\}$ and $\mathcal{I}_2^*=\{0_X, \nu, \lambda\vee\mu, \lambda\wedge\mu, 1_X\}$ on I, and the identity mapping $i_I:(I, \mathcal{I}_1, \mathcal{I}_2)\to(I, \mathcal{I}_1^*, \mathcal{I}_2^*)$, defined by $i_I(x)=x$ for each $x\in I$. It is obvious that i_I is a *fpac* mapping which is not a *fpc* mapping. Also, since 0_X is the only $\mathcal{I}_1-fo$ set contained in ν, $\nu=i_I^{-1}(\nu)$ is not a $(\mathcal{I}_1, \mathcal{I}_2)-fso$ set of $(I, \mathcal{I}_1, \mathcal{I}_2)$ and hence i_I is not a *fpsc* mapping. □

Example 3.6. Let μ_1 and μ_2 be fuzzy sets of $X=\{a, b, c\}$ and let ν be a fuzzy set of $Y=\{x, y\}$, defined as follows;

$\mu_1(a)=0.0,\ \ \mu_1(b)=0.3,\ \ \mu_1(c)=0.2,$
$\mu_2(a)=0.1,\ \ \mu_2(b)=0.0,\ \ \mu_2(c)=0.1,$
$\nu(x)=0.9,\ \ \nu(y)=0.9.$

Consider fuzzy topologies $\mathcal{I}_1=\{0_X, \mu_1, 1_X\}$, $\mathcal{I}_2=\{0_X, \mu_2, 1_X\}$, $\mathcal{I}_1^*=\{0_Y, \nu, 1_Y\}$ and $\mathcal{I}_2^*=\{0_Y, \nu^c, 1_Y\}$. Define $f:(X, \mathcal{I}_1, \mathcal{I}_2)\to(Y, \mathcal{I}_1^*, \mathcal{I}_2^*)$ by $f(a)=y$, $f(b)=f(c)=x$. Then f is a *fpsc* mapping, but f is not a *fpac* mapping. □

Theorem 3.7. Let $f:(X, \mathcal{I}_1, \mathcal{I}_2)\to(Y, \mathcal{I}_1^*, \mathcal{I}_2^*)$ be a mapping. Then the following statements are equivalent:

(i) f is a *fpac* mapping.

(ii) The inverse image of each $(\mathcal{I}_i^*, \mathcal{I}_j^*)-frc$ set of Y is a $\mathcal{I}_i-fc$ set of X.

(iii) $f^{-1}(\nu)\le\mathcal{I}_i-\mathrm{Int}(f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)))$ for each $\mathcal{I}_i-fo$ set ν of Y.

(iv) $\mathcal{I}_i-\mathrm{Cl}(f^{-1}(\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)))\le f^{-1}(\lambda)$ for each $\mathcal{I}_i^*-fc$ set λ of Y.

Proof. (i) implies (ii): Let ν be a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-frc$ set of Y. Then ν^c is a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-fro$ set of Y. Thus $f^{-1}(\nu^c)$ is a $\mathcal{I}_i-fo$ set of X. But $f^{-1}(\nu^c)=(f^{-1}(\nu))^c$. Therefore $f^{-1}(\nu)$ is a $\mathcal{I}_i-fc$ set of X.

(ii) implies (i): Let ν be a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-fro$ set of Y. Then ν^c is a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-frc$ set of Y. Thus $f^{-1}(\nu^c)$ is a $\mathcal{I}_i-fc$ set of X. Therefore $f^{-1}(\nu)$ is a $\mathcal{I}_i-fo$ set of X and consequently, f is a *fpac* mapping.

(i) implies (iii): Let ν be a $\mathcal{I}_i^*-fo$ set of Y. Then $\nu\le\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$. Hence $\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)\le\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)))$.
Now, $\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)\le\mathcal{I}_j^*-\mathrm{Cl}\,\nu$ and

$$\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)))\le\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)).$$

Thus $\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)))\le\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$. This implies that $\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$ is a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-fro$ set of Y. Then $f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu))$ is a $\mathcal{I}_i-fo$ set of X. Therefore

$$\begin{aligned} f^{-1}(\nu)&\le f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)) \\ &=\mathcal{I}_i-\mathrm{Int}(f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu))). \end{aligned}$$

(iii) implies (i): Let ν be a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-fro$ set of Y. Then

$$\begin{aligned} f^{-1}(\nu)&\le\mathcal{I}_i-\mathrm{Int}(f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu))) \\ &=\mathcal{I}_i-\mathrm{Int}(f^{-1}(\nu)). \end{aligned}$$

Thus $f^{-1}(\nu)=\mathcal{I}_i-\mathrm{Int}(f^{-1}(\nu))$. Therefore $f^{-1}(\nu)$ is a $\mathcal{I}_i-fo$ set of X.

(ii) implies (iv): Let λ be a $\mathcal{I}_i^*-fc$ set of Y. Then $\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)\leq\lambda$. Hence

$$\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}(\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)))\leq\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda).$$

Now, $\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)\geq\mathcal{I}_j^*-\mathrm{Int}\,\lambda$, and hence

$$\begin{aligned}\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}(\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)))\\ \geq\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)).\end{aligned}$$

This implies that $\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)$ is a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-frc$ set of Y.
Then $f^{-1}(\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda))$ is a $\mathcal{I}_i-fc$ set of X. Therefore

$$\begin{aligned}\mathcal{I}_i-\mathrm{Cl}(f^{-1}(\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)))=f^{-1}(\mathcal{I}_i-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda))\\ \leq f^{-1}(\lambda).\end{aligned}$$

(iv) implies (ii): Let λ be a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-frc$ set of Y. Then λ is a $\mathcal{I}_i^*-fc$ set of Y.

Hence $\mathcal{I}_i-\mathrm{Cl}(f^{-1}(\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)))\leq f^{-1}(\lambda)$. Since λ is a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-frc$ set of Y, $\mathcal{I}_i^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Int}\,\lambda)=\lambda$ and so $\mathcal{I}_i-\mathrm{Cl}(f^{-1}(\lambda))\leq f^{-1}(\lambda)$. Thus $f^{-1}(\lambda)$ is a $\mathcal{I}_i-fc$ set of X. □

Theorem 3.8. A mapping $f:(X, \mathcal{I}_1, \mathcal{I}_2)\rightarrow(Y, \mathcal{I}_1^*, \mathcal{I}_2^*)$ is *fpac* if and only if there exists a $\mathcal{I}_i-fo$ set μ of X containing x_ω such that $f(\mu)\leq\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$ for each fuzzy point x_ω in X and each $\mathcal{I}_i^*-fo$ set ν of Y containing $f(x_\omega)$.

Proof. Let f be a *fpac* mapping and let x_ω be a fuzzy point in X, and let ν be a $\mathcal{I}_i^*-fo$ set containing $f(x_\omega)$. Then by Theorem 3.7,

$$f^{-1}(\nu)\leq\mathcal{I}_i-\mathrm{Int}(f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu))).$$

Let $\mu=f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu))$ and $\lambda=\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$. Then $\nu\leq\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$. Hence

$$\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)\leq\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu))).$$

Now, $\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)\leq\mathcal{I}_j^*-\mathrm{Cl}\,\nu$ and

$$\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)))\leq\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)).$$

Thus $\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)))\leq\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$. This implies that $\lambda=\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)$ is a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-fro$ set of Y. Therefore $\mu=f^{-1}(\lambda)$ is a $\mathcal{I}_i^*-fo$ set and

$$\begin{aligned}f(\mu)&=f(f^{-1}(\lambda))\\ &=f(f^{-1}(\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)))\\ &\leq\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu).\end{aligned}$$

Conversely, let ν be a $(\mathcal{I}_i^*, \mathcal{I}_j^*)-fro$ set of Y and let x_ω be any fuzzy point in X such that $x_\omega\in f^{-1}(\nu)$. Then there exists a $\mathcal{I}_i-fo$ set μ_{x_ω} of X such that $x_\omega\in\mu_{x_\omega}$ and $f(\mu_{x_\omega})\leq\mathcal{I}_i^*-\mathrm{Int}(\mathcal{I}_j^*-\mathrm{Cl}\,\nu)=\nu$. Thus $x_\omega\in\mu_{x_\omega}\leq f^{-1}(f(\mu_{x_\omega}))\leq f^{-1}(\nu)$. Therefore we have

$$f^{-1}(\nu)=\vee\{x_\omega\,|\,x_\omega\in f^{-1}(\nu)\}\leq\vee\{\mu_{x_\omega}\,|\,x_\omega\in f^{-1}(\nu)\}\leq f^{-1}(\nu).$$

Hence $f^{-1}(\nu)=\vee\{\mu_{x_\omega}\,|\,x_\omega\in f^{-1}(\nu)\}$ and consequently, $f^{-1}(\nu)$ is a $\mathcal{I}_i-fo$ set. □

Theorem 3.9. Let $(X_1, \mathcal{I}_1, \mathcal{I}_2)$, $(X_2, \mathcal{I}_1^*, \mathcal{I}_2^*)$, $(Y_1, \mathcal{G}_1, \mathcal{G}_2)$ and $(Y_2, \mathcal{G}_1^*, \mathcal{G}_2^*)$ be *fbts*'s such that Y_1 is product related to Y_2. Then the product $f_1\times f_2:(X_1\times X_2, \mathcal{M}_1, \mathcal{M}_2)\rightarrow(Y_1\times Y_2, \mathcal{N}_1, \mathcal{N}_2)$, where $\mathcal{M}_k$ (respectively $\mathcal{N}_k$) is the fuzzy product topology, generated by $\mathcal{I}_k$ and $\mathcal{I}_k^*$ (respectively $\mathcal{G}_k$ and $\mathcal{G}_k^*$) $(k=1, 2)$, of *fpac* mappings $f_1:(X_1, \mathcal{I}_1, \mathcal{I}_2)\rightarrow(Y_1, \mathcal{G}_1, \mathcal{G}_2)$ and $f_2:(X_2, \mathcal{I}_1^*, \mathcal{I}_2^*)\rightarrow(Y_2, \mathcal{G}_1^*, \mathcal{G}_2^*)$, is a *fpac* mapping.

Proof. Let $\lambda=\vee(\mu_m\times\nu_n)$, where μ_m's are $\mathcal{G}_i-fo$ sets of Y_1 and ν_n's are $\mathcal{G}_i^*-fo$ sets of Y_2, be an $\mathcal{N}_i-fo$ sets of $Y_1\times Y_2$. Then, we have

$$\begin{aligned}&(f_1\times f_2)^{-1}(\lambda)\\ &=\vee(f_1^{-1}(\mu_m)\times(f_2^{-1}(\nu_n))\\ &\leq\vee(\mathcal{I}_i-\mathrm{Int}(f_1^{-1}[\mathcal{G}_i-\mathrm{Int}(\mathcal{G}_j-\mathrm{Cl}\,\mu_m)])\\ &\quad\times\mathcal{I}_i^*-\mathrm{Int}(f_2^{-1}[\mathcal{G}_i^*-\mathrm{Int}(\mathcal{G}_j^*-\mathrm{Cl}\,\nu_n)]))\end{aligned}$$