On the Stability Issues of Linear Takagi-Sugeno Fuzzy Models

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ABSTRACT

Stability issues of linear Takagi-Sugeno fuzzy models are thoroughly investigated. At first, a systematic way of searching for a common symmetric positive definite P matrix (common P matrix in short), which is related to stability, is proposed for N subsystems which are under a pairwise commutativity assumption. Robustness issue under modeling uncertainty in each subsystem is then considered by proposing a quadratic stability criterion and a method of determining uncertainty bounds. Finally, it is shown that the pairwise commutative assumption can be in fact relaxed by interpreting the uncertainties as mismatch parts of non-commutative system matrices. Several examples show the validity of the proposed methods.

I. Introduction

Fuzzy logic has been considered as an efficient and effective tool in managing uncertainties of systems since Zadeh's seminal paper [Zadeh [1]]. Among many applications of fuzzy logic, control design appears to be one that has attracted a large amount of attention in the past two decades.

In general, fuzzy control system can be classified as Mamdani type and Takagi-Sugeno (T-S in short) type. In spite of certain qualitative analysis, the Mamdani type fuzzy control system is still generally recognized as empirically-based. One major reason of this is the lack of successful fuzzy models useful for the Mamdani type fuzzy control system. On the other hand, the T-S type fuzzy control system mainly focuses on this subject (see, e.g., Takagi and Sugeno [2]).

There are some work in literature that are mainly concerned with the stability analysis of T-S fuzzy model. The existence of a proper T-S fuzzy model is first assumed. Tanaka and Sugeno [3] showed that the stability of a T-S fuzzy model could be shown by finding a common symmetric positive definite matrix P for N subsystems. This has been considered a very important result and some refining efforts have been pursued thereafter. There has not been, however, a systematic way to find the common P matrix in a general framework. Kawamoto *et al.* [4] only considered a 2nd order system. Tanaka [5] suggested the idea of using LMI (Linear Matrix Inequality) for finding the common P matrix. Xia and Chai [6] proposed a stability condition which is based on *ad hoc* membership values. Zhao *et al.* [7] extended some past work to consider uncertainty.

The current work endeavors to tackle the stability issue of the T-S fuzzy model by first considering the problem as related to the "switching system" [Fu and Barmish [8] and Narendra and Balakrishnan [9, 10]]. In a sense, the central issue lies in the search for a *common positive definite matrix P for multiple matrix equations*. Then uncertainty is introduced into the set-

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ting and robustness analysis for stability is performed. It is shown that a somewhat stringent assumption for the common matrix P in fact can be relaxed.

The main contributions of the paper are threefold. First, an iterative algorithm for the choice of a common P matrix is proposed. It is shown that the algorithm can indeed reach a solution for systems under a structural assumption. Second, modeling uncertainty of T-S fuzzy model is introduced. A robustness analysis on the influence of uncertainty toward stability is also suggested. The analysis is non-conservative and computationally straightforward. Third, it is shown that the structural assumption can be in fact relaxed by interpreting the uncertainties as mismatch parts of system matrices. This in turn means this algorithm is applicable to both the uncertainty and non-structured case.

Ⅱ. Background Materials

2.1 Takagi and Sugeno's Fuzzy Model

Takagi and Sugeno [2] proposed an effective way to represent a fuzzy model of a dynamical system. It uses a linear input-output relation as its consequence of individual plant rules[1]. A T-S fuzzy model is composed of N plant rules that can be represented as

Rule i: *if* $x_1(k)$ *is* M_1^i *and* $x_2(k)$ *is* M_2^i *and* $\cdots$ *and* $x_n(k)$ *is* M_n^i
then $x(k+1)=A_i x(k)+B_i u(k),\ i=1,2,\cdots,N$ (1)

where

k: discrete time index,
x_j: j^{th} state (or linguistic) variable,
M_j^i: a fuzzy term of M_j selected for *Rule i*,
M_j: fuzzy term set[2] of x_j,
$x(k)$: state vector and $x(k)=\left[x_1(k)\ x_2(k)\cdots x_n(k)\right]^T\in R^n$,
$u(k)$: input vector and $u(k)=\left[u_1(k)\ u_2(k)\cdots u_m(k)\right]^T\in R^m$,
$A_i\in R^{n\times n}$,
$B_i\in R^{n\times m}$.

For any current state vector $x(k)$ and input vector $u(k)$, the T-S fuzzy model infers $x(k+1)$ as the output of the fuzzy model as follows:

$$x(k+1)=\frac{\sum_{i=1}^{N} w_i(k)\left[A_i x(k)+B_i u(k)\right]}{\sum_{j=1}^{N} w_j(k)} \tag{2}$$

where

$$w_i(k)=\prod_{j=1}^{N} M_j^i(x_j(k)). \tag{3}$$

For a free system (i.e., $u(k)\equiv 0$), eq. (2) can be written as

$$x(k+1)=\frac{\sum_{i=1}^{N} w_i(k)\ A_i\ x(k)}{\sum_{j=1}^{N} w_j(k)}. \tag{4}$$

It is assumed, from now on, a proper T-S fuzzy model in the form of eq. (4) is available.

2.2 Formulation of the Stability Problem

Tanaka and Sugeno [3] suggested an important criterion for the stability of the T-S fuzzy model.

Theorem 1. [Tanaka and Sugeno [3]]
The equilibrium of the fuzzy system (4) (namely, $x=0$) is globally asymptotically stable if there exists a common symmetric positive definite matrix P such that

$$A_i^T P A_i - P < 0 \ \textit{for all}\ i=1,2,\cdots,N. \tag{5}$$

So far, however, no systematic way of finding the common P exists. It is suggested to translate the stability problem of T-S fuzzy model to be one for the following N-simultaneous linear systems:

$$x_i(k+1)=A_i x_i(k),\ i=1,2,\cdots,N \tag{6}$$

where the system matrix A_i for the i^{th} plant rule is the same as eq. (1).

Ⅲ. Stability Analysis of Linear T-S Fuzzy Model

Narendra and Balakrishnan [10] suggested a systematic way of finding the common P matrix of N simultaneous continuous-time linear systems under a pairwise commutativity assumption. The results are now extended to discrete-time systems.

At first, it is considerd $N=2$ for simplicity. Let a T-S fuzzy model be with two plant rules, i.e., A_1 and A_2.

Theorem 2. Suppose that A_1 and A_2 are Hurwitz and commutative, i.e., $A_1A_2=A_2A_1$. Consider the following two Lyapunov equations

$$A_1^TP_1A_1-P_1=-Q \tag{7}$$
$$A_2^TP_2A_2-P_2=-P_1 \tag{8}$$

where $Q>0$ and P_1 and P_2 are the unique symmetric positive definite solutions of eqs. (7) and (8), respectively. Then we always have

$$A_i^TP_2A_i-P_2<0,\ i=1,2. \tag{9}$$

Proof: Substituting the P_1 in eq. (8) into eq. (7) yields

$$\begin{aligned}-Q&=A_1^T(-A_2^TP_2A_2+P_2)A_1+A_2^TP_2A_2-P_2\\&=-A_1^TA_2^TP_2A_2A_1+A_1^TP_2A_1+A_2^TP_2A_2-P_2\end{aligned} \tag{10}$$

Since A_1 and A_2 are commutative, we have $(A_1, A_2)^T$ and hence $A_2^TA_1^T=A_1^TA_2^T$. Therefore, eq. (10) becomes

$$\begin{aligned}-Q&=-A_2^TA_1^TP_2A_1A_2+A_1^TP_2A_1+A_2^TP_2A_2-P_2\\&=A_2^T(-A_1^TP_2A_1)A_2+A_1^TP_2A_1+A_2^TP_2A_2-P_2\\&=A_2^T(-A_1^TP_2A_1+P_2)A_2-(-A_1^TP_2A_1+P_2)\\&=A_2^T\Psi_1A_2-\Psi_1\end{aligned} \tag{11}$$

where

$$\Psi_1:=-A_1^TP_2A_1+P_2. \tag{12}$$

Since $Q>0$ and A_2 is Hurwitz, the "solution" and is unique.
By choosing

$$V(x(k))=x^T(k)P_2x(k) \tag{13}$$

for both systems, we have, for the A_2 system,

$$\begin{aligned}\Delta V&=x^T(k)\,(A_2^TP_2A_2-P_2)x(k)\\&=-x^T(k)\,P_1x(k)<0,\ \forall\ x(k)\neq 0\end{aligned} \tag{14}$$

since $P_1>0$ from eq. (7). In addition,

$$\begin{aligned}\Delta V&=x^T(k)\,(A_1^TP_2A_1-P_2)x(k)\\&=-x^T(k)\Psi_1\,x(k)<0,\ \forall\ x(k)\neq 0\end{aligned} \tag{15}$$

for the A_1 system since $\Psi_1>0$. Eqs. (14) and (15) show that $A_i^TP_2A_i-P_2<0$ for $i=1, 2$.

(End of Proof)

Theorem 1 shows that P_2 can be used as the common P matrix of the T-S fuzzy model with two plant rules. The technique can be extended to a T-S fuzzy model with N plant rules, i.e., $A_1, A_2,\ldots, A_N$.

Theorem 3. Suppose that A_i is Hurwitz for all $i=1, 2,\ldots, N$ and A_i's are pairwise commutative, i.e.,

$$A_jA_{j+1}=A_{j+1}A_j,\ j=1,2,\cdots,N-1. \tag{16}$$

Consider the following N Lyapunov equations

$$A_1^TP_1A_1-P_1=-Q$$

$$A_2^T P_2 A_2 - P_2 = -P_1$$
$$\vdots \qquad (17)$$
$$A_N^T P_N A_N - P_N = -P_{N-1}$$

where $Q > 0$ and P_i, $i = 1, 2, \ldots, N$ is the unique symmetric positive definite solution of each equation. Then we always have

$$A_i^T P_N A_i - P_N < 0,\ i = 1, 2, \ldots, N \qquad (18)$$

Proof: The proof follows the procedure similar to that of Theorem 2.

(End of Proof)

Remark 1: The choice of the order of A_i is apparently non-unique. One may in practice attempt to label A_i in the most appropriate way in order that the pairwise commutativity is achieved.

The following example illustrates the use of Theorem 3.

(Example 1) Consider a T-S fuzzy model with three plant rules and two state variables. Let the corresponding A_i's be

$$A_1 = \begin{bmatrix} 1.0 & 0.2 \\ -0.3 & 0.5 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.5 & 0.12 \\ -0.18 & 0.2 \end{bmatrix},\ \text{and } A_3 = \begin{bmatrix} 0.8 & 0.16 \\ -0.24 & 0.4 \end{bmatrix}.$$

It can be easily seen that they are all Hurwitz. Furthermore, they are pairwise commutative since

$$A_1 A_2 = A_2 A_1 = \begin{bmatrix} 0.464 & 0.16 \\ -0.24 & 0.064 \end{bmatrix} \text{ and}$$

$$A_2 A_3 = A_3 A_2 = \begin{bmatrix} 0.3712 & 0.1280 \\ -0.192 & 0.0512 \end{bmatrix}.$$

Let $Q = I$, we then have the following:

$$A_1^T P_1 A_1 - P_1 = -Q \text{ yields } P_1 = \begin{bmatrix} 7.3084 & 2.0053 \\ 2.0053 & 2.2579 \end{bmatrix},$$

$$A_2^T P_2 A_2 - P_2 = -P_1 \text{ yields } P_2 = \begin{bmatrix} 9.2163 & 2.6734 \\ 2.6734 & 2.6239 \end{bmatrix},$$

$$A_3^T P_3 A_3 - P_3 = -P_3 \text{ yields } P_3 = \begin{bmatrix} 19.3746 & 6.5438 \\ 6.5438 & 4.7113 \end{bmatrix}.$$

The P_3 is symmetric and positive definite since the corresponding eigenvalues are 21.8702 and 2.2157. Now, let us check the legitimacy of P_3 to be the common P matrix.

Table 1. Common Matrix P_3

$A_i^T P_3 A_i - P_3$	Eigenvalues	$A_i^T P_3 A_i - P_3 < 0$
i = 1	-3.6160 and -1.3360	satisfied
i = 2	-17.4354 and -2.0505	satisfied
i = 3	-10.1642 and -1.6760	satisfied

Table 1 shows that P_3 is indeed a common P matrix of the T-S fuzzy model.

(Q.E.D.)

In summary, a common symmetric positive definite P matrix which guarantees the stability of a T-S fuzzy model with N plant rules can be found systematically from Theorem 3. It consists of solving N Lyapunov equations iteratively from an arbitrary symmetric $Q > 0$. The resulting P_N is indeed a common symmetric positive definite P matrix. The symmetric positive definite matrix P_N is obtained under the assumption of Hurwitz and pairwise commutative system matrices.

Ⅳ. Quadratic Stability Analysis of Linear T-S Fuzzy Model

A T-S fuzzy model composed of N uncertain plant rules can be represented as follows:

Rule i: if $x_1(k)$ *is* M_1^i *and* $x_2(k)$ *is* M_2^i *and* $\cdots$ *and* $x_n(k)$ *is* M_n^i
then $x(k+1) = [A_i + \Delta A_i(k)]x(k),\ i = 1, 2, \cdots, N$ (19)

where A_i's are the nominal system matrices and the corresponding $\Delta A_i(k)$'s are (possibly) time-varying uncertainties. It can be translated to

$$x_i(k+1)=\left[A_i+\Delta A_i(k)\right]x_i(k),\ i=1,2,\cdots,N \tag{20}$$

for stability analysis like eq. (6).

For systems under (possibly) time-varying uncertainties, quadratic stability performance can be used as a basis for stability study. The quadratic stability performance can be readily extended to the systems in eq. (20). The following definition is proposed.

Definition 1. Consider following system with N uncertain plant rules

$$x_i(k+1)=\left[A_i+\Delta A_i(k)\right]x_i(k),\ i=1,2,\cdots,N$$

where $\Delta A_i(k)\in\Omega_i$ and Ω_i is a compact set. The uncertain system is quadratically stable if there exists a common symmetric positive definite matrix P and a common positive constant $\underline{\lambda}$ such that

$$x^T\left[(A_i+E_i)^T P(A_i+E_i)-P\right]x\le -\underline{\lambda}\|x\|^2,\ i=1,2,\cdots,N \tag{21}$$

for all $E_i\in\Omega_i$ and $x\in R^n$ for all $i=\{1,2,\cdots,N\}$.

Checking quadratic stability of a T-S fuzzy model in (20) using the Definition 1 directly may be sometimes difficult, especially for high dimensional case. Therefore, a more realistic way to use Definition 1 is required. One may expect to use the nominal system only since the uncertainty is unknown anyway. In Section 3, a common P matrix (i.e., P_N) is chosen systematically by way of Theorem 3 under the pairwise commutativity assumption. Motivated by this, the P_N can be used as a reasonable choice of the common P in eq. (21) with the pairwise commutativity assumption of nominal system matrices. The following proposition is made.

Proposition 1. The T-S fuzzy model (20) is quadratically stable if there exists a positive constant $\underline{\lambda}$ such that

$$x^T\left[(A_i+E_i)^T P_N(A_i+E_i)-P_N\right]x\le -\underline{\lambda}\|x\|^2,\ i=1,2,\cdots,N \tag{22}$$

for all $E_i\in\Omega_i$ (Ω_i's are compact sets) and $x\in R^n$, $i=\{1,2,\cdots,N\}$. Here A_i's are Hurwitz and pairwise commutative and P_N is chosen from the corresponding nominal system by way of Theorem 3.

Corollary 1. A T-S fuzzy model in eq. (20) is quadratically stable if

$$\max_{E_i\in\Omega_i}\lambda_{\max}\left[(A_i+E_i)^T P_N(A_i+E_i)-P_N\right]<0 \tag{23}$$

holds for arbitrary $E_i\in\Omega_i$ (Ω_i's are compact), $i=\{1,2,\cdots,N\}$, where A_i's are Hurwitz and pairwise commutative and P_N is a common symmetric positive definite matrix. Here $\lambda_{\max}[\cdot]$ denotes the maximum eigenvalue of the designated symmetric matrix.

Even though (23) is more explicit than (22), it is still difficult to apply it in practice. Corollary 1 can be more tractable by adding an assumption to Ω, i.e., Ω is convex set. It is shown in Gu et *al.* [12] under the convexity assumption, the maximum in (26) for $N=1$ can be reached by one of the protruded points (i.e., vertices) of the set Ω. The additional convexity assumption of Ω does not degrade seriously the generality of uncertainty since elements of E in Ω can often be represented by scalar bounds, i.e., $|e_i(l,m)|\le d_i(k)$ where $e_i(l,m)$ denotes the (l,m) element of E_i and $d_i(k)$ denotes a possibly time-varying scalar bound. Therefore, a theorem which makes the Corollary 1 to be more tractable can be made for a T-S fuzzy model with N uncertain plant rules using the convexity assumption as follows.

Theorem 4. Consider the maximization problem

$$\max_{E_i\in\Omega_i}\lambda_{\max}\left[(A_i+E_i)^T P_N(A_i+E_i)-P_N\right] \tag{24}$$

where P_N is a fixed symmetric positive definite matrix and Ω_i's ($i=1, 2, \cdots, N$) are compact and convex sets for the i^{th} plant rule. The maximum for each i can be reached by one of the protruded points of each set Ω_i.

Remark 2: Corollary 1 and Theorem 4 suggest that only protruded points are needed to be checked for quadratic stability.

The following example illustrates the use of Corollary 4.1 and Theorem 4.1.

(Example 2) Consider a T-S fuzzy model in Example 1. The system matrices are

$$A_1=\begin{bmatrix} 1.0 & 0.2 \\ -0.3 & 0.5 \end{bmatrix},\ A_2=\begin{bmatrix} 0.5 & 0.12 \\ -0.18 & 0.2 \end{bmatrix},\ \text{and}\ A_3=\begin{bmatrix} 0.8 & 0.16 \\ -0.24 & 0.4 \end{bmatrix}.$$

They are Hurwitz and pairwise commutative as shown in Example 1. For simplicity, let the uncertainty sets for each plant rules be given as

$$\Delta A_1(k)=\Delta A_2(k)=\Delta A_3(k)=\begin{bmatrix} d_1(k) & d_2(k) \\ d_2(k) & d_1(k) \end{bmatrix}$$

where $d_1(k)$ and $d_2(k)$ are uncertainty parameters satisfying $|d_1(k)| \le \rho$ and $|d_2(k)| \le 2\rho$ with $\rho=0.02$. Then the uncertainty sets Ω_1, Ω_2, and Ω_3 become

$$\Omega_1=\Omega_2=\Omega_3=\begin{bmatrix} \{d_1 \| d_1 | \le \rho\} & \{d_2 \| d_2 | \le 2\rho\} \\ \{d_2 \| d_2 | \le 2\rho\} & \{d_1 \| d_1 | \le \rho\} \end{bmatrix} \quad (25)$$

Eq. (25) shows that Ω_1, Ω_2, and Ω_3 are convex and there are four protruded points in each uncertainty sets. They are

$$E_1=\begin{bmatrix} -\rho & -2\rho \\ -2\rho & -\rho \end{bmatrix},\ E_2=\begin{bmatrix} -\rho & 2\rho \\ 2\rho & -\rho \end{bmatrix},$$
$$E_3=\begin{bmatrix} \rho & -2\rho \\ -2\rho & \rho \end{bmatrix},\ E_4=\begin{bmatrix} \rho & 2\rho \\ 2\rho & \rho \end{bmatrix}. \quad (26)$$

Quadratic stability of the system can be checked using Corollary 1 and Theorem 4. As in Example 1, the common symmetric positive definite matrix P_3 is

$$P_3=\begin{bmatrix} 19.3746 & 6.5438 \\ 6.5438 & 4.7113 \end{bmatrix}$$

by $Q=I$. Corollary 1 and Theorem 4 yield

Tables 2~4 show that the uncertain system is quadratically stable by Proposition 1.

(Q.E.D.)

Remark 3: It is shown that the quadratic stability in Proposition 1 can be checked by using Theorem 4 under the assumption of compact and convex uncertainty set. It means that $\underline{\lambda}$ in Proposition 4 can be determined from the result of Theorem 4 as follows:

$$\underline{\lambda}=\min_i\{\underline{\lambda}_i\} \quad (27)$$

where $\underline{\lambda}_i$'s are determined from

$$x^T\left[(A_i+E_i)^T P_N (A_i+E_i)-P_N\right]x \le -\underline{\lambda}_i\|x\|^2. \quad (28)$$

Table 2. First Plant Rule A_1

E_i	Eigenvalues of $\left[(A_1+E_i)^T P_3 (A_1+E_i)-P_3\right]$	$\lambda_{max}\left[(A_1+E_i)^T P_3 (A_1+E_i)-P_3\right]$
E_1	-5.3344 and -1.3737	-1.3737
E_2	-3.7867 and -0.9989	-0.9989
E_3	-3.8249 and -1.2390	-1.2390
E_4	-2.5762 and -0.4814	-0.4814

Table 3. Second Plant Rule A_2

E_i	Eigenvalues of $\left[(A_2+E_i)^T P_3 (A_2+E_i) - P_3\right]$	$\lambda_{\max}\left[(A_2+E_i)^T P_3 (A_2+E_i) - P_3\right]$
E_1	-18.2529 and -2.0580	-2.0580
E_2	-17.3527 and -2.0309	-2.0309
E_3	-17.5272 and -2.0067	-2.0067
E_4	-16.5355 and -1.9873	-1.9873

Table 4. Third Plant Rule A_3

E_i	Eigenvalues of $\left[(A_3+E_i)^T P_3 (A_3+E_i) - P_3\right]$	$\lambda_{\max}\left[(A_3+E_i)^T P_3 (A_3+E_i) - P_3\right]$
E_1	-11.5460 and -1.6853	-1.6853
E_2	-10.0617 and -1.6400	-1.6400
E_3	-10.3318 and -1.5925	-1.5925
E_4	-8.7684 and -1.5425	-1.5425

In summary, the symmetric positive definite common matrix P_N can be used for the quadratic stability analysis of N uncertain subsystems which have Hurwitz and pairwise commutative system matrices. With the assumption of compact and convex hyperpolyhedron uncertainty set, the quadratic stability of T-S fuzzy model with uncertainties can be checked easily. Therefore, the common matrix P_N can indeed play more general role than Narendra and Balakrishnan [10].

V. Analysis of Uncertainty Bounds of Linear T-S Fuzzy Model

Section 4 considers the quadratic stability of a general T-S fuzzy model with N pairwise commutative nominal system matrices. The uncertainty sets Ω_i's are fixed, i.e., the uncertainty bounds are specified in Section 4. It is, however, also interesting to finding the maximum allowable uncertainty bounds for the T-S fuzzy model in eq. (20) using P_N as a common P matrix. The problem can be formulated as follows.

Problem 1 : Consider a T-S fuzzy model with N uncertain plant rules as

$$x_i(k+1) = [A_i + \Delta A_i(k)]x_i(k),\ i \in \{1, 2, \cdots, N\}$$

where A_i's are Hurwitz and pairwise commutative and ΔA_i's are (possibly) time-varying uncertainties. Find the maximum possible sets $\Lambda_1, \Lambda_2, \cdots \Lambda_N$ such that

$$x^T\left[(A_i+E_i)^T P_N (A_i+E_i) - P_N\right]x \leq -\underline{\lambda}\|x\|^2,\ i \in \{1, 2, \cdots, N\}$$

holds for arbitrary $E_i \in \Lambda_i$ where P_N is the common symmetric positive definite matrix determined from the corresponding nominal system and $\underline{\lambda}$ is a positive constant.

Gu [13] proposed an algorithm calculating the maximum allowable uncertainty bound for the system in (20) with N = 1 under the additional assumption that Ω has fixed shape but "adjustable" size. A new uncertainty set Ω_δ with the additional assumption can be stated as follows. Let

$$\Omega_\delta = \{ \delta E \mid E \in \Omega_{fixed} \} \tag{29}$$

where Ω_{fixed} is a compact and convex set with fixed shape and size, δ is constant. The constant δ is shown to be non-negative in Gu [13] in the case of stable nominal system. By relating Ω_δ to the uncertainty bound, eq. (29) means that the uncertainty bound is represented as scalar multiple to a fixed prototype of uncertainty bound. It is solvable and quite general since many problems can be formulated in this way by choosing appropriate fixed uncertainty set Ω_{fixed}. The uncertainty set in eq. (29) can be extended to N systems by defining

$$\Omega_{\delta_i} = \{ \delta_i E_{fixed_i} \mid E_{fixed_i} \in \Omega_{fixed_i} \} \tag{30}$$

where the subscript i^{th} denotes the ith plant rule. Problem 1 can be restated as follows using the uncertainty set in eq. (30).

Problem 2: Consider a T-S fuzzy model with N uncertain plant rules as

$$x_i(k+1) = [A_i + \Delta A_i(k)] x_i(k),\ i \in \{1, 2, \cdots, N\}$$

where A_i's are Hurwitz and pairwise commutative, $\Delta A_i(k) \in \Omega_{\delta_i}$. Here is defined as in eq. (30). Find the maximum possible sets $\Lambda_1, \Lambda_2, \cdots, \Lambda_N$ with

$$\Lambda_i = \{ (\delta_i)_{\max} E_{fixed_i} \mid E_{fixed_i} \in \Omega_{fixed_i} \} \tag{31}$$

such that (20) is quadratically stable where $(\delta_i)_{\max}$ denotes the maximum value of δ_i.

From the pairwise commutativity of A_i's and Corollary 1, Problem 2 can be reduced to finding δ_i such that

$$\max_{E_{fixed_i} \in \Omega_{fixed_i}} \lambda_{\max} \left[\left(A_i + \delta_i E_{fixed_i}\right)^T P_N \left(A_i + \delta_i E_{fixed_i}\right) - P_N \right] < 0 \tag{32}$$

for every i, $i = 1, 2, \cdots, N$.

Gu [13] suggested an algorithm of finding the maximum possible uncertainty bound of the system in (20) with N = 1 by using the uncertainty set defined in eq. (29). It can be extended to N systems in Problem 2 to determine the maximum possible bound of each plant rules of T-S fuzzy model in (32) and the following proposition holds.

Proposition 5.1. Consider the T-S fuzzy model with N uncertain plant rules as

$$x_i(k+1) = [A_i + \Delta A_i(k)] x_i(k),\ i \in \{1, 2, \cdots, N\}$$

where A_i's are Hurwitz and pairwise commutative and $\Delta A_i(k) \in \Omega_{\delta_i}$,

$$\Omega_{\delta_i} = \{ \delta_i E_{fixed_i} \mid E_{fixed_i} \in \Omega_{fixed_i} \}.$$

Suppose that Ω_{fixed_i}'s are fixed sets which are compact and convex hyperpolyhedron. Then the maximum possible bounds of $\Delta A_i(k)$'s defined by

$$\Lambda_i = \{ (\delta_i)_{\max} E_{fixed_i} \mid E_{fixed_i} \in \Omega_{fixed_i} \},\ i = \{1, 2, \cdots, N\}$$

can be determined by solving the following

$$\frac{1}{\delta_i^*(E_{fixed_i})} = \max \left\{ \beta_i \mid \lambda_{\max} \left[\begin{bmatrix} 0 & -E_{fixed_i} \\ -E_{fixed_i}^T & 0 \end{bmatrix} \xi_i = \beta_i \begin{bmatrix} P_N^{-1} & A_i \\ A_i^T & P_N \end{bmatrix} \xi_i \right] = 0 \right\} \tag{33}$$

$$\frac{1}{(\delta_i)_{\max}} = \max_{E_{fixed_i} \in \Omega_{fixed_i}} \frac{1}{\delta_i^*(E_{fixed_i})},\ i \in \{1, 2, \cdots, N\} \tag{34}$$

via the finite protruded points of each Ω_{fixed_i}'s.

Here $\beta_i = \frac{1}{\delta_i}$.

The following example illustrates the use of Proposition 2.

(Example 3) Example 2 shall be revisited with some modifications. Suppose that

$$\Omega_{fixed_i} = \{ E^1_{fixed_i}, E^2_{fixed_i}, E^3_{fixed_i}, E^4_{fixed_i} \}$$

where

$$E^1_{fixed_i} = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} E^2_{fixed_i} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$E^3_{fixed_i} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} E^4_{fixed_i} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

For simplicity, it is assumed that $\Omega_{fixed_1} = \Omega_{fixed_2} = \Omega_{fixed_3}$. The following results are obtained from eqs. (33) and (34):

Tables 5~7 show that the maximum eigenvalues for each system matrices A_1, A_2, and A_3 are 38.8201, 8.0488, and 14.3584, respectively. This in turn means that the maximum uncertainty bounds for each plant rules are their reciprocals, i.e., 0.0258, 0.1242, and 0.0696, respectively. These can be interpreted as the bounds of tolerance when the nominal system is under the pairwise commutativity assumption.

(Q.E.D.)

Table 5. First Plant Rule A_1

$E^j_{fixed_1}$	Eigenvalues β_i	Maximum eigenvalue
$E^1_{fixed_1}$	-38.8201, 6.9381, -0.4060, 0.8256	6.9381
$E^2_{fixed_1}$	19.7271, -15.0050, 0.6970, -0.4376	19.7271
$E^3_{fixed_1}$	-19.7271, 15.0050, -0.6970, 0.4376	15.0050
$E^4_{fixed_1}$	38.8201, -6.9381, 0.4060, -0.8256	38.8201

Table 6. Second Plant Rule A_2

$E^j_{fixed_1}$	Eigenvalues β_i	Maximum eigenvalue
$E^1_{fixed_1}$	-8.0488, 4.5479, -0.4801, 0.6941	4.5479
$E^2_{fixed_1}$	6.0621, -6.2352, 0.6347, -0.5085	6.0621
$E^3_{fixed_1}$	-6.0621, 6.2352, -0.6347, 0.5085	6.2352
$E^4_{fixed_1}$	8.0488, -4.5479, 0.4801, -0.6941	8.0488

Table 7. Third Plant Rule A_3

$E^j_{fixed_1}$	Eigenvalues β_i	Maximum eigenvalue
$E^1_{fixed_1}$	-14.3584, 5.3170, -0.4322, 0.7759	5.3170
$E^2_{fixed_1}$	8.8935, -9.1748, 0.6800, -0.4814	8.8935
$E^3_{fixed_1}$	-8.8935, 9.1748, -0.6800, 0.4814	9.1748
$E^4_{fixed_1}$	14.3584, -5.3170, 0.4322, -0.7759	14.3584

Example 3 can be checked indirectly by reconsidering the results of Example 4.2. The uncertainty bound ρ was found to be 0.02 in Example 2. It is smaller than the maximum allowable bound 0.0258 in Example 3. Therefore, it can be reasonably explained why the system in Example 2 is quadratically stable. The greater maximum eigenvalues for A_2 and A_3 in Example 2 can be explained by the greater maximum allowable bounds of A_2 and A_3 in Example 3.

Remark 3: In this section, analyses are focused on the T-S fuzzy model with uncertainties which has Hurwitz and pairwise commutative nominal system matrices. One may, however, interpret it as non-Hurwitz and/or non-pairwise commutative system matrices without uncertainties. It means that the Hurwitz and pairwise commutativity assumption in Section 3 can be relaxed by interpreting the uncertainties flexibly. Consider the following system

$$x_i(k+1) = A_i x_i(k) \ i \in \{1, 2, \cdots, N\}$$

where A_i's are non-Hurwitz and/or pairwise non-commutative. One can consider the decomposition of A_i's such as

$$A_i = \overline{A}_i + \Delta A_i, \ i \in \{1, 2, \cdots, N\} \tag{35}$$

where

$$\overline{A}_j \overline{A}_{j+1} = \overline{A}_{j+1} \overline{A}_j, \ j \in \{1, 2, \cdots, N-1\} \tag{36}$$

The original system can be rewritten as follows:

$$x_i(k+1) = \left[\overline{A}_i + \Delta A_i\right] x_i(k), \ i \in \{1, 2, \cdots, N\} \tag{37}$$

where $\overline{A}_i$'s are Hurwitz and pairwise commutative. The "mismatched" portion $\Delta \overline{A}_i$'s can be treated as uncertainties as in Section 4. Therefore the pairwise commutativity assumption in Section 3 can be relaxed.

In summary, uncertainty bounds of T-S fuzzy model can be determined from the symmetric positive definite matrix P_N which is obtained from Hurwitz and pairwise commutative system matrices. The uncertainty bounds are represented as a scalar multiplier of fixed prototypes of uncertainties Ω_{fixed_i}'s. In the case of convex hyperpolyhedron Ω_{fixed_i}'s, the problem is limited to the search of finite protruded points. Furthermore, the uncertainties can be interpreted more flexibly in order to relax the Hurwitz and pairwise commutative A-matrices assumption in Section 3.

Ⅵ. Conclusions

Stability issues of linear T-S fuzzy model is investigated thoroughly under the assumption of pairwise commutative system matrices. At first, a systematic way of finding a common symmetric positive matrix

P is proposed. It can be determined analytically from any $Q>0$ and P_N is revealed to be the common P matrix. The common P matrix can be used for the stability analysis of linear T-S fuzzy model with uncertainties. The problem of quadratic stability of a linear T-S fuzzy model with N uncertain plant rules where the corresponding nominal system matrices are pairwise commutative is then investigated. A quadratic stability criterion for T-S fuzzy model is proposed. A method of finding uncertainty bounds is also investigated using the common P matrix. Several examples verify the validity of the proposed methods. Finally, the possibility of relaxation of the pairwise commutativity assumption is mentioned by interpreting the uncertainties as mismatch parts of non-commutative system matrices.

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