Notes on Fuzzy Equivalence Relations

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ABSTRACT

In this paper we define the t-fuzzy equivalence relation on a set and we prove some properties in connection with t-fuzzy relations.

I. Introduction

Since Zadeh[6] have introduced the definition of a fuzzy relation from X to Y as a fuzzy subset of $X \times Y$, the theory of fuzzy relations was developed by [3, 4, 5]. In [4], Nemitz has studied lattice-valued fuzzy relations assuming values from a Brouwerian lattice. In [5], Sidky defined the t-fuzzy partition on a set and studied its properties.

In this paper we define the notion of t-fuzzy relation on a set and proved some related properties. In details, our main results are:

1. For every t-fuzzy equivalence relation on X, let the fuzzy subset $P_{[x]_{\overline{R^t}}} : X \to I$ be defined by

$$P_{[x]_{\overline{R^t}}}(z) = \begin{cases} \vee_{R(x,z)>t} R(x, z) & \text{if } z \in [x]_{\overline{R^t}} \\ 0 & \text{if } z \notin [x]_{\overline{R^t}} \end{cases}$$

   for all $x \in X$. Then $\{P_{[x]_{\overline{R^t}}} \mid x \in X\}$ is a t-fuzzy partition of X.

2. For every t-fuzzy partition of X, $P = \{P_i \in F(X) \mid i \in I\}$, the fuzzy subset $R : X \times X \to I$ by

$$R(x, y) = \vee_i P_i(x) \wedge P_i(y), \qquad (x, y) \in X \times X$$

   is a t-fuzzy equivalence relation on X.

3. Let R and S be t-fuzzy equivalence relations on X as a complete Brouwerian lattice [1]-fuzzy subset of $X \times X$ respectively.
   Then $R \circ S$ is a t-fuzzy equivalence relation on X iff $R \circ S = S \circ R$

II. Preliminaries

Let L denote a linear lattice with the universal bounds 0, 1 and $t \in L - \{0\}$ throughout this paper.

Definition 2.1. [5]. (1) Let X be a nonempty set and L be a lattice. An L-fuzzy subset A of X is defined to be a mapping $A : X \to L$.
(2) Let $F(X) = \{A \mid A \text{ is an } L\text{-fuzzy subset of } X\}$.

Definition 2.2. [5] Let A be an L-fuzzy subset of X and $t \in L$. The subset $\overline{A^t} = \{x \in X \mid A(x) > t\}$ of X is called a strong t-level subset of X.

Definition 2.3. [5] Let X and Y be nonempty sets. A fuzzy relation from X to Y is defined to be an L-fuzzy subset of $X \times Y$.

Definition 2.4. [5] If R is a fuzzy relation from X to Y i.e, $R \in F(X \times Y)$, then the fuzzy relation $R^{-1} \in F(Y \times$

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X) defined by $R^{-1}(y, x)=R(x, y)$, for all $x\in X$ and $y\in Y$, is called the inverse relation of R.

Definition 2.5. [4] Let $R\in F(X\times Y)$ and $S\in F(Y\times Z)$. The max-min composition of R and S is defined as a fuzzy relation $S\circ R$ from X to Z such that

$S\circ R(x, z)=\vee_y R(x, y)\wedge S(y, z),\ \forall(x, z)\in X\times Z.$

Definition 2.6. [4] Let X be a nonempty set and $R\in F(X\times Y)$, R is called an L-valued fuzzy equivalence relation on X iff

(1) R is reflexive i.e, $R(x, x)=1$, $\forall x\in X$.

(2) R is symmetric i.e, $R^{-1}=R$.

(3) R is transitive i.e, $R\circ R\subseteq R$.

Definition 2.7. [5] Let X be a nonempty set. The class $P=\{Y_i\in P(X)\,|\,i\in I\}$ is called a t-fuzzy partition of X iff

(1) for each $x\in X$, there is $Y_i\in P$ such that $Y_i(x)>t$.

(2) for each $Y_i\in P$, there is $x\in X$ such that $Y_i(x)>t$.

(3) if $i\neq j$, then $Y_i(x)\wedge Y_j(x)\leq t$, $\forall x\in X$.

Ⅲ. Main Results

Definition 3.1. Let $R\in F(X\times X)$. R is called a t-fuzzy equivalence relation on X iff

(1) R is t-reflexive i.e. $R(x, x)>t$, $\forall x\in X$.

(2) $R^{-1}=R$.

(3) $R\circ R\subseteq R$.

Theoreme 3.2. Let $R\in F(X\times X)$ be a t-fuzzy equivalence relation on X. Then $\overline{R^t}=\{(x, y)\,|\,R(x, y)>t\}$ is an ordinary equivalence relation on X.

Proof. Firstly, for each $x\in X$, since R is t-reflexive, $R(x, x)>t$, and so $(x, x)\in\overline{R^t}$, which implies $\overline{R^t}$ is reflexive.

Secondly, let $(x, y)\in\overline{R^t}$. Then $R(x, y)>t$. Since R is symmetric, $R(y, x)=R^{-1}(x, y)>t$, and so $(y, x)=\overline{R^t}$, which implies $\overline{R^t}$ is symmetric. Finally, let $(x, y)\in\overline{G^t}$, and $(y, z)\in\overline{R^t}$. Then $R(x, y)>t$ and $R(y, z)>t$. By transitivity of R, $R(x, z)>t$, and so $(x, z)\in\overline{R^t}$, which implies $\overline{R^t}$ is transitive. Therefore, $\overline{R^t}$ is an ordinary equivalence relation on X.

Theorem 3.3. Let $R\in F(X\times X)$ is a t-fuzzy equivalence relation on X, and let $[x]_{\overline{R^t}}=\{y\in X\,|\,R(x, y)>t\}$, $x\in X$

Then $\{[x]_{\overline{R^t}}\,|\,x\in X\}$ is an ordinary partition of X.

Proof. For each $x\in X$, since R is t-reflexive, $R(x, x)>t$ and so $x\in[x]_{\overline{R^t}}$, which implies $X\subseteq\bigcup_{x\in X}[x]_{\overline{R^t}}$.

We claim that $[x]_{\overline{R^t}}=[y]_{\overline{R^t}}$ if $[x]_{\overline{R^t}}\cap[y]_{\overline{R^t}}\neq\phi$. Suppose that $[x]_{\overline{R^t}}\cap[y]_{\overline{R^t}}\neq\phi$, then there exists $z\in X$ such that $z\in[x]_{\overline{R^t}}\cap[y]_{\overline{R^t}}$. This implies that $R(x, z)>t$ and $R(y, z)>t$.

By transitivity of R, $R(x, y)>t$. If $z'\in[x]_{\overline{R^t}}$, then $R(x, z')>t$. Combining $R(x, y)>t$ and $R(z', x)>t$, which implies $R(z', y)>t$, hence $[x]_{\overline{R^t}}\subseteq[y]_{\overline{R^t}}$. Similiarly, interchanging the roles of x and y, we get that $[y]_{\overline{R^t}}\subseteq[x]_{\overline{R^t}}$. Therefore, $[x]_{\overline{R^t}}\subseteq[y]_{\overline{R^t}}$.

Definition 3.4. Let $R\in F(X\times X)$ be a t-fuzzy equivalence relation on X. For each $x\in X$, we define a fuzzy subset $P_{[x]_{\overline{R^t}}}: X\to I$ by

$$P_{[x]_{\overline{R^t}}}(z)=\begin{cases}\vee_{R(x,z)>t}R(x, z) & \text{if } z\in[x]_{\overline{R^t}}\\ 0 & \text{if } z\notin[x]_{\overline{R^t}}\end{cases}$$

Theorem 3.5. If $[x]_{\overline{R^t}}\cap[y]_{\overline{R^t}}\neq\phi$, then $P_{[x]_{\overline{R^t}}}(z)=P_{[y]_{\overline{R^t}}}(z)$, $z\in X$.

Proof. Let $y\in[x]_{\overline{R^t}}$. We show that $P_{[x]_{\overline{R^t}}}(z)=P_{[y]_{\overline{R^t}}}(z)$, $z\in X$. If $z\in[x]_{\overline{R^t}}$, then $R(x, z)>t$. By transivity of R, $z\in[y]_{\overline{R^t}}$ which implies $P_{[x]_{\overline{R^t}}}(z)=P_{[y]_{\overline{R^t}}}(z)$. Otherwise, since $y\in[x]_{\overline{R^t}}$, $z\notin[y]_{\overline{R^t}}$, which implies $P_{[x]_{\overline{R^t}}}(z)=0=P_{[y]_{\overline{R^t}}}(z)$. Therefore, $P_{[x]_{\overline{R^t}}}(z)=P_{[y]_{\overline{R^t}}}(z)$ for all $z\in X$.

Theorem 3.6. $P=\{P_{[x]_{\overline{R^t}}}\,|\,x\in X\}$ is a t-fuzzy partition of X.

Proof. Firstly, for each $z\in X$, Since R is t-reflexive $R(z, z)>t$, which implies $P_{[z]_{\overline{R^t}}}(z)\geqq R(z, z)>t$.

Secondly, for each $P_{[z]_{\overline{R^t}}}\in P$, we have $P_{[z]_{\overline{R^t}}}(z)>t$.

Finally, we show that $[x]_{\overline{R^t}} \cap [y]_{\overline{R^t}} \neq \phi$ implies $P_{[x]_{\overline{R^t}}}(z) \wedge P_{[y]_{\overline{R^t}}}(z) \leqq t$, $z \in X$. If $z \in [x]_{\overline{R^t}}$, since $\{[x]_{\overline{R^t}} \mid x \in X\}$ is an ordinary partition of X, then $z \notin [y]_{\overline{R^t}}$, and so $P_{[y]_{\overline{R^t}}}(z) = 0$, which implies $P_{[x]_{\overline{R^t}}}(z) \wedge P_{[z]_{\overline{R^t}}}(z) = 0$. If $z \in [y]_{\overline{R^t}}$, then $z \notin [x]_{\overline{R^t}}$, and so $P_{[x]_{\overline{R^t}}}(z) = 0$, which implies $P_{[x]_{\overline{R^t}}}(z) \wedge P_{[y]_{\overline{R^t}}}(z) = 0$. If $z \notin [x]_{\overline{R^t}}$ and $z \notin [y]_{\overline{R^t}}$, then $P_{[x]_{\overline{R^t}}}(z) = 0 = P_{[y]_{\overline{R^t}}}(z)$, and so $P_{[x]_{\overline{R^t}}}(z) \wedge P_{[y]_{\overline{R^t}}}(z) = 0$. In which case, we have $P_{[x]_{\overline{R^t}}}(z) \wedge P_{[y]_{\overline{R^t}}}(z) \leqq t$ for all $z \in X$.

Theorem 3.7. Let $P = \{P_i \in F(X) \mid i \in I\}$ be a t-fuzzy partition of X. Then there exists the correspondent t-fuzzy equivalence relation on X.

Proof. Define a fuzzy subset $R: X \times X \to I$ by

$$R(x, y) = \vee_i P_i(x) \wedge P_i(y), \qquad (x, y) \in X \times X.$$

We show that R is a t-fuzzy equivalence relation on X. Firstly, let $x \in X$. Since P is a t-fuzzy partition of X, there exists a $P_i \in P$ such that $P_i(x) > t$. This implies that $R(x, x) = \vee_i P_i(x) \wedge P_i(x) = \vee_i P_i(x) \geqq P_i(x) > t$. Therefore, R is t-reflexive.

Secondly, by definition, $R(x, y) = R(y, x)$.

$$\begin{aligned} \text{Finally,} \quad R \circ R(x, y) &= \vee_{z \in X} R(x, z) \wedge R(z, y) \\ &= \vee_{z \in X}(\vee_i(P_i(x) \wedge P_i(z)) \\ &\quad \wedge \vee_i(P_i(z) \wedge P_i(y))) \\ &\leqq \vee_{z \in X}(\vee_i(P_i(x) \wedge P_i(z) \wedge P_i(y)) \\ &\leqq \vee_i \vee_{z \in X}(P_i(x) \wedge P_i(y)) \\ &= \vee_i(P_i(x) \wedge P_i(y)) = R(x, y) \\ &\quad \text{for all } (x, y) \in X \times X. \end{aligned}$$

Therefore, R is transitive. Consequently, R is a t-equivalence relation on X.

Theorem 3.8. Let R be a t-fuzzy equivalence relation on X and let $t > s$, $0 \leqq s < 1$. Then R is a s-fuzzy equivalence relation on X, and $P_t \subseteq P_s$, where $P_t = \{P_{[x]_{\overline{R^t}}} \mid x \in X\}$, $P_s = \{P_{[x]_{\overline{R^s}}} \mid x \in X\}$.

Proof. To show that R is a s-fuzzy equivalence relation on X, it suffices to show that R is s-reflexive. For each $x \in X$, since R is t-reflexive, $R(x, x) > t$, which implies $R(x, x) > s$. Therefore, R is s-reflexive. The rest is clear. If $y \in [x]_{\overline{R^t}}$, then $R(x, y) > t$. Since $t > s$, which implies $R(x, y) > s$ and so $y \in [x]_{\overline{R^s}}$, therefore, $[x]_{\overline{R^t}} \subseteq [x]_{\overline{R^s}}$. Consequently, we have $P_t \subseteq P_s$.

Theorem 3.9. Let R and S be t-fuzzy equivalence relations on X as a complete brouwerian lattice [1]-fuzzy subset of $X \times X$ respectively.

Then $R \circ S$ is a t-fuzzy equivalence relation on R iff $R \circ S = S \circ R$

Proof. Suppose that $R \circ S$ is a t-fuzzy equivalence relation on X. Since R, S and $R \circ S$ are symmetric respectively, we have $R \circ S = (R \circ S)^{-1} = S^{-1} \circ R^{-1} = S \circ R$. Conversely, suppose that $R \circ S = S \circ R$. We show that $R \circ S$ is a t-fuzzy equivalence relation on X. Firstly let $x \in X$. Since R and S are t-reflexive respectively, $R(x, x) > t$, and $S(x, x) > t$, and so $(R \circ S)(x, x) = \vee_y S(x, y) \wedge R(y, x) \geqq R(x, x) \wedge S(x, x) > t$ which implies $R \circ S$ is t-reflexive. Secondly, by hypothesis, $R \circ S = S \circ R = (R \circ S)^{-1}$. Finally, we show that $R \circ S$ is transitive. For this consider

$$\begin{aligned} (R \circ S) \circ (R \circ S) &= R \circ (S \circ R) \circ S \quad \text{by lemma 2.7 of Sidky [5]} \\ &= R \circ (R \circ S) \circ S \\ &\subseteq R \circ S. \end{aligned}$$

Therefore, $R \circ S$ is transitive. This completes the proof.

Corollary 3.10. Let R be a t-fuzzy equivalence relation on X. Then $R \circ R$ is t-fuzzy equivalence relation on X.

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