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On the Effect of Estimated Mean for the Weighted Symmetric Estimator

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Abstract

The ordinary least squares estimator and the corresponding pivotal statistics have been widely used for the unit root test. Recently several test criteria based on maximum likelihood estimators and weighted symmetric estimator have been proposed for testing the unit root hypothesis in the autoregressive processes. Pantula *at el.* (1994) showed that the weighted symmetric estimator has good power properties. In this article we use an adjusted estimator for mean in the model when we use weighted symmetric estimator. A simulation study shows that for the small samples, this new test criterion has better power properties than the weighted symmetric estimator.

1. Introduction

Consider the first order autoregressive process given a forward representation

$$Y_t - \mu = \rho(Y_{t-1} - \mu) + a_t$$

and a backward representation

$$Y_t - \mu = \rho(Y_{t+1} - \mu) + \eta_t$$

where $\{a_t\}, \{\eta_t\}$ are sequences of serially uncorrelated $(0, \sigma^2)$ random variables.

Now consider the class of estimators, where the estimator of ρ is the ρ that minimizes

$$Q(\rho) = \sum_{t=2}^{n} w_t [Y_t - \mu - \rho(Y_{t-1} - \mu)]^2 + \sum_{t=1}^{n-1} (1 - w_{t+1}')[Y_t - \mu - \rho(Y_{t+1} - \mu)]^2. \qquad (1.1)$$

The ordinary least squares estimator studied by Dickey and Fuller (1979) is obtained by setting $w_t = 1$. The estimator obtained by setting $w_t = 0.5$ is the symmetric estimator studied by Dickey *at el.* (1984). The weighted symmetric estimator is constructed with

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$w_t = \frac{t-1}{n}$, $t = 1, 2, 3, \cdots n$. For more details, see Fuller (1996).

For this model, the unit root test is a test of $H_0 : \rho = 1$ vs $H_1 : \rho < 1$. These three estimators can be used for the unit root test. The weighted symmetric estimator has the best power properties among them. See the results reported by Pantula *at el.* (1994). Also the limiting distributions can be found in Pantula *at el.* (1994)

For the zero mean case, the limiting distribution of the weighted symmetric estimator defined by $\hat{\rho}_{use} = \{\sum_{t=2}^{n} Y_t Y_{t-1}\}\{\sum_{t=2}^{n-1} Y_t^2 + \sum_{t=1}^{n} Y_t^2/n\}^{-1}$ is

$$n(\hat{\rho}_{use} - 1) \rightarrow^L (2G)^{-1}(T^2 - 1 - 2G),$$

where $(G, T) = \sum_{i=1}^{\infty}(g_i^2 a_i^2, \sqrt{2} g_i a_i)$, $g_i = (-1)^{i+1} 2\{(2i-1)\pi\}^{-1}$ and $\{a_i\}$ is a sequence of independent normal random variables with mean zero and variance one. The pivotal statistic which is similar to the usual t-test is given by

$$\hat{\tau}_{use} = \{\hat{\rho}_{use} - 1\}\{\sum_{t=2}^{n-1} Y_t^2 + \sum_{t=1}^{n} Y_t^2/n\}^{1/2} \cdot s_{use}^{-1}$$

and the limiting distribution is

$$\hat{\tau}_{use} \rightarrow^L (2G)^{-1/2}(T^2 - 1 - 2G),$$

where $s_{use}^2 = \frac{1}{n-2} Q(\hat{\rho}_{\mu, use})$.

For the mean estimated case, the limiting distribution of the weighted symmetric estimator is known as following:

$$n(\hat{\rho}_{\mu, use} - 1) \rightarrow^L (G - H^2)^{-1}\{(T^2 - 1)/2 - TH + 2H^2 - G\}, \tag{1.2}$$

where $\hat{\rho}_{\mu, use} = \frac{\sum_{t=2}^{n}(Y_t - \hat{\mu})(Y_{t-1} - \hat{\mu})}{\sum_{t=2}^{n-1}(Y_t - \hat{\mu})^2 + \sum_{t=1}^{n}(Y_t - \hat{\mu})^2/n}$, $\hat{\mu} = \frac{\sum_{t=1}^{n-1} Y_t + \sum_{t=2}^{n} Y_t}{(2n-2)}$ and $H = \sum_{i=1}^{\infty} 2^{1/2} g_i^2 a_i$.

The pivotal weighted symmetric statistic for the mean estimated case is

$$\hat{\tau}_{\mu, use} = \frac{\hat{\rho}_{\mu, use} - 1}{\{\sum_{t=2}^{n-1}(Y_t - \hat{\mu})^2 + \sum_{t=1}^{n}(Y_t - \hat{\mu})^2/n\}^{-1/2} \cdot s_{use}} \tag{1.3}$$

and the limiting distribution is

$$\hat{\tau}_{use} \rightarrow^L (G - H^2)^{-1/2}\{(T^2 - 1)/2 - TH + 2H^2 - G\}.$$

A time series with zero mean is seldom encountered in practice. Except for the ordinary least squares estimator, the pivotal estimators generally have power that is comparable to or greater than that based on $n(\hat{\rho}-1)$. Thus in this paper we focus on the pivotal estimator for the mean estimated case. The empirical distribution of the pivotal weighted symmetric statistic is tabulated in Fuller(1996).

In section 2 we develop an adjusted weighted symmetric estimator and section 3 contains the empirical distributions and the powers of the proposed estimator. We have made some concluding remarks in section 4.

2. Adjusted weighted symmetric estimator.

In this section we develop an adjusted mean estimator for the weighted symmetric estimator. First consider the estimators of mean μ such as

$$\widehat{\mu}_1=\sum_{t=1}^{n}\frac{Y_t}{n}, \tag{2.1}$$

$$\widehat{\mu}_s=\frac{\sum_{t=1}^{n-1}Y_t+\sum_{t=2}^{n}Y_t}{(2n-2)}. \tag{2.2}$$

Fuller (1996) indicated that the power of using (2.2) is better than that of using (2.1) for the small sample cases. Define an alternative estimator of mean as following:

$$\widehat{\mu}_a=\frac{\sum_{t=1}^{n}Y_t+2(Y_1-Y_n)}{n}. \tag{2.3}$$

where Y_1 and Y_n are the first and the last observations respectively. Note that in the unit root case the order of Y_n is $O_p(\sqrt{n})$.

The proposed adjusted weighted symmetric estimator is

$$\hat{\rho}_{\mu,ause}=\frac{\sum_{t=2}^{n}(Y_t-\widehat{\mu}_a)(Y_{t-1}-\widehat{\mu}_s)}{\sum_{t=2}^{n-1}(Y_t-\widehat{\mu}_s)^2+\sum_{t=1}^{n}(Y_t-\widehat{\mu}_s)^2/n}$$

and the pivotal estimator is

$$\hat{\tau}_{\mu,ause}=\frac{\hat{\rho}_{\mu,ause}-1}{\{\sum_{t=2}^{n-1}(Y_t-\widehat{\mu}_s)^2+\sum_{t=1}^{n}(Y_t-\widehat{\mu}_s)^2/n\}^{-1/2}\cdot s_{use}}, \tag{2.4}$$

where $s_{use}^2=\frac{1}{n-2}Q(\hat{\rho}_{\mu,ause})$.

Due to the effect of the first observation and the last one in the unit root case, we shall show that the test criterion based on (2.4) has better power than that of Fuller(1996) in (1.3).

3. Simulation results

In this section, we compare the empirical power of the two test criteria described in (1.3) and (2.4). Here we have in mind the test of $\rho=1$ against the alternative that $\rho<1$. We first present the empirical distributions of the test based on the adjusted weighted symmetric estimator and then study empirical powers.

3.1 Empirical Percentiles

Our model is $Y_t-\mu=\rho(Y_{t-1}-\mu)+a_t$, where $a_t \sim N(0,1)$. To construct the percentiles, we let $Y_1=a_1$, $\mu=0$, $\rho=1$, and generate the a_t as independent standard normal random variables. The RNNOR function in FORTRAN is used to generate the a_t's. For a given sample size n, we generated 50,000 replications of sample size n and computed the test statistics. The percentiles are reported in Tables 1 and Table 2.

Table 1. Percentiles of $\hat{\tau}_{\mu,ause}$, the pivotal adjusted weighted symmetric estimator

Sample size	Probability of a Smaller Value							
	0.01	0.025	0.05	0.1	0.9	0.95	0.975	0.99
25	-3.34	-2.92	-2.58	-2.22	0.77	1.29	1.73	2.21
50	-3.19	-2.84	-2.55	-2.27	0.39	0.84	1.23	1.66
100	-3.16	-2.82	-2.53	-2.23	0.19	0.59	0.93	1.37
250	-3.14	-2.81	-2.52	-2.22	0.06	0.42	0.74	1.19

Table 2. Empirical 5% Critical Values for Unit Root Test

Test statistic		n			
		25	50	100	250
$\hat{\tau}_{\mu,wse}$	Pantula(1994)	-2.66	-2.61	-2.58	-2.54
	Fuller(1996)	-2.60	-2.57	-2.55	-2.54
	Proposed	-2.66	-2.59	-2.56	-2.53
$\hat{\tau}_{\mu,ause}$		-2.58	-2.55	-2.53	-2.52

We used the proposed critical values for the comparison of the power of $\hat{\tau}_{\mu.awse}$. Notice that the used critical values are slightly different from Pantula(1994) and Fuller(1996).

3.2 Empirical powers

In this section we study the empirical power of the statistics described in (1.3) and (2.4). Critical values for 5%-level tests are given in Table 2. But the 5%-level critical values are slightly different from that reported in Fuller(1996) and Pantula(1994). In this comparison we used the critical values reported in Table 2. We considered two cases, (1) $Y_1 \sim N(0,1)$ and (2) $Y_1 \sim N[0,(1-\rho^2)^{-1}]$ and generated samples of size n=25, 50, 100, and 250 with $\rho=0.98$, 0.95, 0.90, 0.85, 0.80, and 0.70. The powers are based on 50,000 Monte Carlo replications. Empirical powers are summarized in the following Table 3 to Table 6. In case (1) we used $Y_1 = a_1$, whereas $Y_1 = a_1(1-\rho^2)^{-1/2}$ in case (2).

Table 3. Empirical powers for 5%-level Test criteria(n=25, 50,000 replications)

	ρ						
Statistic	0.98	0.95	0.93	0.90	0.85	0.8	0.7
				$Y_1 \sim N(0,1)$			
$\hat{\tau}_{\mu.wse}$	6.21	7.98	9.43	12.06	16.91	23.05	39.21
$\hat{\tau}_{\mu.awse}$	6.31	8.13	9.78	12.51	17.65	24.11	40.48
				$Y_1 \sim N[0,(1-\rho^2)^{-1}]$			
$\hat{\tau}_{\mu.wse}$	5.43	6.65	7.74	9.90	13.88	19.11	35.23
$\hat{\tau}_{\mu.awse}$	5.53	6.73	7.98	10.01	14.28	19.61	35.53

Table 4. Empirical powers for 5%-level Test criteria(n=50, 50,000 replications)

	ρ						
Statistic	0.98	0.95	0.93	0.90	0.85	0.8	0.7
				$Y_1 \sim N(0,1)$			
$\hat{\tau}_{\mu.wse}$	7.63	12.97	17.39	25.01	41.50	59.73	87.92
$\hat{\tau}_{\mu.awse}$	7.71	13.14	17.68	25.51	42.33	60.52	88.09
				$Y_1 \sim N[0,(1-\rho^2)^{-1}]$			
$\hat{\tau}_{\mu.wse}$	6.43	9.97	13.27	19.70	34.38	53.25	84.95
$\hat{\tau}_{\mu.awse}$	6.45	10.02	13.38	19.80	34.55	53.15	84.58

Table 5. Empirical powers for 5%-level Test criteria(n=100, 50,000 replications)

Statistic	ρ = 0.98	0.95	0.93	0.90	0.85	0.8	0.7
				$Y_1 \sim N(0,1)$			
$\hat{\tau}_{\mu,wse}$	11.26	25.91	39.38	61.49	88.74	98.32	99.99
$\hat{\tau}_{\mu,awse}$	11.37	26.30	40.01	62.30	89.16	98.34	99.99
				$Y_1 \sim N[0,(1-\rho^2)^{-1}]$			
$\hat{\tau}_{\mu,wse}$	8.60	19.64	30.78	52.81	83.90	96.90	99.97
$\hat{\tau}_{\mu,awse}$	8.66	19.77	31.00	53.00	83.88	96.75	99.95

Table 6. Empirical powers for 5%-level Test criteria(n=250, 50,000 replications)

Statistic	ρ = 0.98	0.95	0.93	0.90	0.85	0.8	0.7
				$Y_1 \sim N(0,1)$			
$\hat{\tau}_{\mu,wse}$	26.77	78.69	95.82	99.85	100.00	100.00	100.00
$\hat{\tau}_{\mu,awse}$	26.95	79.01	95.91	99.85	100.00	100.00	100.00
				$Y_1 \sim N[0,(1-\rho^2)^{-1}]$			
$\hat{\tau}_{\mu,wse}$	20.05	69.99	92.46	99.64	99.99	100.00	100.00
$\hat{\tau}_{\mu,awse}$	20.11	70.01	92.38	99.61	99.99	100.00	100.00

By the results of Table 3 to Table 6, we observe the followings.

(1) $Y_1 \sim N(0,1)$

The test criterion based on $\hat{\tau}_{\mu,awse}$, the adjusted weighted symmetric estimator, is superior to that of $\hat{\tau}_{\mu,wse}$, the weighted symmetric estimator, for all sample sizes.

(2) $Y_1 \sim N[0,(1-\rho^2)^{-1}]$

1. For sample size n=25, the test criterion based on $\hat{\tau}_{\mu,awse}$ is superior to that of $\hat{\tau}_{\mu,wse}$.

2. For sample size n=50, 100 and ρ close to 1, the test criterion based on $\hat{\tau}_{\mu,awse}$ is superior to that of $\hat{\tau}_{\mu,wse}$. For values of ρ smaller than 0.8, the test criterion based on $\hat{\tau}_{\mu,awse}$ is competitive to that of $\hat{\tau}_{\mu,wse}$.

3. For sample size n=250, the test criteria based on $\hat{\tau}_{\mu, wse}$ and $\hat{\tau}_{\mu, awse}$ have the almost same powers for all values of ρ.

4. Concluding remarks

We have discussed two test criteria for testing the null hypothesis of a unit root in AR(1) process. Based on our simulation studies, regardless of initial value assumption, the criterion of the adjusted weighted symmetric estimator is superior to that of the weighted symmetric estimator when the sample size is smaller than 100. For the large sample size case, the two test criteria achieved almost the same powers for all values of ρ, which verifies that the two test criterion have the same limiting distribution.

Even though we do not study the other criteria such as the criteria suggested by Elliot *at el.*(1992), the criteria of maximum likelihood estimator by Gonzalez-Farias (1992), comparing the simulation results reported by Pantula *at el.* (1994) and ours, the criterion of the adjusted weighted symmetric estimators has fairy good power properties when sample size is smaller than 100.

So this paper gives us some evidences to improve the weighted symmetric estimator for the unit root case in the small sample case and for the small sample case, this new test criterion is recommended.

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