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The Partial Ordering of Positive Lower Orthant Dependence[1)]

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Abstract

In this note we develop a partial ordering among positive lower orthant dependent distributions with fixed marginals. This permits us to measure the degree of positive lower orthant dependence. Some basic properties and preservation results are derived.

1. Introduction

Let $\underline{X} = (X_1, \cdots, X_n)$ be a random vector. It is said to be positively upper orthant dependent ($PUOD$) if for every $\underline{x} = (x_1, \cdots, x_n)$ $P(\underline{X} > \underline{x}) \geq \Pi_{i=1}^{n} P(X_i > x_i)$ and it is said to be positively lower orthant dependent ($PLOD$) if for every $\underline{x} = (x_1, \cdots, x_n)$ $P(\underline{X} \leq \underline{x}) \geq \Pi_{i=1}^{n} P(X_i \leq x_i)$. The random vector $\underline{X}$ is said to be positively orthant dependent (POD) if $\underline{X}$ is $PUOD$ and $PLOD$ (Ahmed et al.(1978)). The Positive dependence has been continuously examined by many authors. See Block and Ting(1981), Chhetry, Kimeldorf and Sampson(1989) and Barlow and Proschan(1981). By way of some motivations, we have to compare the degree of positive dependence of two sets of positive lower orthant dependent random vectors. In bivariate case, Ahmed et al.(1978) have already studied very extensively the partial ordering of positive quadrant dependence. Ebahimi(1982) has also introduced the partial ordering of negative quadrant dependence.

In this note we introduce the notions of the positive lower orthant dependence ordering, and derive some basic properties and preservation results. Before concluding this section, we introduce the concept of positive quadrant dependence ordering, which be useful in what follows.

Let $\beta_1 = \beta_1(F, G)$ denote the class of bivariate distribution functions(df's) H on R^2 having specified marginal df's F and G where both F and G are nondegenerate.

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Definition 1.1(Lehmann, 1996) The pair (X, Y) or its distribution H is positively quadrant dependent (PQD) if

$$P(X \leq x, Y \leq y) \geq P(X \leq x) P(Y \leq y) \text{ for all } (x, y) \in R^2. \tag{1.1}$$

Let $\overline{\beta_1}$ denote the subclass of β_1 where H is PQD. Suppose H_1 and H_2 both belong to $\overline{\beta_1}$.

Definition 1.2(Ahmed et al., 1979) The bivariate distribution H_1(or random vector $\underline{X}$) is said to be more positively quadrant dependent than H_2 (or random vector $\underline{Y}$) if

$$H_1(x, y) \geq H_2(x, y) \text{ for all } (x, y) \in R^2. \tag{1.2}$$

We write $H_1 \geq^{PQD} H_2$ (or $\underline{X} \geq^{PQD} \underline{Y}$).

2. Some Properties

Let $\beta = \beta(F_1, \cdots, F_n)$ denote the class of $n-$variate distribution functions(df's) H on R^n having marginals $F_1, \cdots, F_n$. Let $\overline{\beta}$ denote the subclass of β where H is $PLOD$.

Definition 2.1 Suppose H_1 and H_2 both belong to $\overline{\beta}$. The distribution H_1 (or random vector $\underline{X}$) is more positively lower orthant dependent than H_2 (or random vector $\underline{Y}$) if

$$H_1(c_1, \cdots, c_n) \geq H_2(c_1, \cdots, c_n) \text{ for all } (c_1, \cdots, c_n) \in R^n. \tag{2.1}$$

We write $H_1 \geq^{PLOD} H_2$ (or $\underline{X} \geq^{PLOD} \underline{Y}$).

Example 2.2 Let $H_0(x_1, \cdots, x_n) = \Pi_{i=1}^n F_i(x_i)$ and $H^*(x_1, \cdots, x_n) = \Lambda_{i=1}^n F_i(x_i)$, where $\Lambda_{i=1}^n F_i(x_i) = \min(F_1(x_1), \cdots, F_n(x_n))$. Define

$$H_\alpha = (1-\alpha) H_0 + \alpha H^*, \ 0 \leq \alpha \leq 1. \tag{2.2}$$

Then H_0, H^* and H_α beling to $\overline{\beta}$ and $H_0 \leq^{PLOD} \leq H_\alpha \leq^{PLOD} H^*$(see Section 4).

Example 2.3 Consider a Farlie-Morgenstern system with distribution

$$\begin{aligned} H_\alpha(x_1, x_2, x_3) &= H_0(x_1, x_2, x_3)[1 + \alpha\{(1-F_1(x_1))(1-F_2(x_2)) \\ &+ (1-F_2(x_2))(1-F_3(x_3)) + (1-F_1(x_1))(1-F_3(x_3)) \\ &+ (1-F_1(x_1))(1-F_2(x_2))(1-F_3(x_3))\}] \end{aligned} \tag{2.3}$$

where $0 < \alpha < 1$, and $H_0(x_1, x_2, x_3) = F_1(x_1)F(x_2)F_3(x_3)$ (See [7]). Then H_0, $H_\alpha \in \overline{\beta}$ and $H_\alpha \geq^{PLOD} H_0$.

Remark 2.4 In (2.3) if $0 < \alpha_1 < \alpha_2 < 1$ then $H_{\alpha_1} \leq^{PLOD} H_{\alpha_2}$.

Example 2.5 Let $\underline{X} \sim \underline{U} + V\underline{e}$ where $\underline{U} = (U_1, \cdots, U_n)$, $\underline{e} = (1, \cdots, 1)$ and V and U_i, $i = 1, \cdots, n$ are independent random variables having distributions $U_i \sim N(0, \sigma^2)$, $V \sim N(0, \delta^2)$. Then $\underline{X} \sim N(0, \sigma^2 I + \delta^2 \underline{e}'\underline{e})$. By adjusting all the one dimensional marginals as $N(0, 1)$ $N(0, \sigma^2 I + \delta^2 \underline{e}'\underline{e})$ is written as

$$N_p(0, (1-p)I + p\underline{e}'\underline{e}), 0 \leq p < 1. \tag{2.4}$$

It is obvious that $N_p(0, (1-p)I + p\underline{e}'\underline{e})$ is $PLOD$ and that

$$N_p(0, (1-p)I + p\underline{e}'\underline{e}) \geq^{PLOD} N(0, I).$$

Remark 2.6 In (2.4) if $0 \leq p_1 < p_2 < 1$ then $N_{p_1} \leq^{PLOD} N_{p_2}$.

Proposition 2.7 Let $\underline{X} = (X_1, \cdots, X_n)$ and $\underline{Y} = (Y_1, \cdots, Y_n)$ be two n-dimensional random vectors with distribution functions F and G, respectively.

$$F(x_1, \cdots, x_n) \geq G(x_1, \cdots, x_n) \text{ for all } \underline{x} = (x_1, \cdots, x_n) \in R^n$$

if and only if

$$E\{I_L(\underline{X})\} \geq E\{I_L(\underline{Y})\} \text{ for all lower orthants } L, \tag{2.5}$$

where the lower orthants L are the sets of the form $\{\underline{x}: x_1 \leq a_1, \cdots, x_n \leq a_n\}$ for some fixed $\underline{a} = (a_1, \cdots, a_n)$.

Theorem 2.8 Let the distribution H_1 of $\underline{X} = (X_1, \cdots, X_n)$ and the distribution H_2 of $\underline{Y} = (Y_1, \cdots, Y_n)$ belong to $\overline{\beta}$. Then $\underline{X} \geq^{PLOD} \underline{Y}$ if and only if

$$E\{\Pi_{i=1}^n h_i(X_i)\} \geq E\{\Pi_{i=1}^n h_i(Y_i)\} \tag{2.6}$$

for every collection $\{h_1, \cdots, h_n\}$ of univariate nonnegative decreasing functions.

Proof. ($\Rightarrow$) Assume $\underline{X} \geq^{PLOD} \underline{Y}$. Let ϕ be an n-variate function of the form

$$\phi(x_1, \cdots, x_n) = \Pi_{i=1}^n h_i(x_i), \ \underline{x} \in R^n,$$

where the h_i's are univariate nonnegative decreasing functions. Every such function can be approximated by positive linear combinations of indicator functions of lower orthants. Thus using (2.5) we obtain (2.6).

($\Leftarrow$) Assume (2.6) holds. By taking $h_i(X_i) = I_{[X_i \leq a_i]}$ and $h_i(Y_i) = I_{[Y_i \leq a_i]}$ we have

$$E[\Pi_{i=1}^n I_{[X_i \leq a_i]}] \geq E[\Pi_{i=1}^n I_{[Y_i \leq a_i]}]$$

$$\Rightarrow E[I_{[X_1 \leq a_1, \cdots, X_n \leq a_n]}] \geq E[I_{[Y_1 \leq a_1, \cdots, Y_n \leq a_n]}]$$

$$\Rightarrow P(X_1 \leq a_1, \cdots, X_n \leq a_n) \geq P(Y_1 \leq a_1, \cdots, Y_n \leq a_n).$$

Thus $\underline{X} \geq^{PLOD} \underline{Y}$ and the proof completes.

From Theorem 2.8 we obtain the following example:

Example 2.9 Let $\underline{X}=(X_1,\cdots,X_n)$ and $\underline{Y}=(Y_1,\cdots,Y_n)$ be two nonnegative random vectors. If $\underline{X}\geq^{PLOD}\underline{Y}$ then according to (2.6) we have

$$E\{\exp(-s\sum_{i=1}^{n}a_iX_i)\}\geq E\{\exp(-s\sum_{i=1}^{n}a_iY_i)\} \text{ for all } s\geq 0,$$

whenever $a_i\geq 0$, $i=1,\cdots,n$ since $\exp(-sa_ix_i)$ are nonegative decreasing functions.

Theorem 2.10 Let $\underline{X}$ and $\underline{Y}$ be two nonnegative $n-$dimensional random vectors and let both the distribution H_1 of $\underline{X}$ and the distribution H_2 of $\underline{Y}$ belong to $\overline{\beta}$. Then $X\geq^{PLOD}\underline{Y}$ if and only if for all s

$$P(\max\{a_1X_1,\cdots,a_nX_n\}\leq s)\geq P(\max\{a_1Y_1,\cdots,a_nY_n\}\leq s) \tag{2.7}$$

whenever $a_i>0$, $i=1,\cdots,n$.

Proof. ($\Rightarrow$) Assume $X\geq^{PLOD}\underline{Y}$. Then for $a_i>0$, $i=1,\cdots,n$

$$P(a_1X_1\leq s,\cdots,a_nX_n\leq s)\geq P(a_1Y_1\leq s,\cdots,a_nY_n\leq s)$$

which yields

$$P(\max\{a_1X_1,\cdots,a_nX_n\}\leq s)\geq P(\max\{a_1Y_1\leq s,\cdots,a_nY_n\}\leq s).$$

($\Leftarrow$) Assume that (2.7) holds. Then for $a_i>0$

$$P(a_1X_1\leq s,\cdots,a_nX_n\leq s)\geq P(a_1Y_1\leq s,\cdots,a_nY_n\leq s)$$

which implies

$$P(X_1\leq c_1,\cdots,X_n\leq c_n)\geq P(Y_1\leq c_1,\cdots,Y_n\leq c_n) \text{ for all } (c_1,\cdots,c_n)\in R^n.$$

Thus the proof is complete.

We now pay our attention to a simple but important property of the class $\overline{\beta}$.

Theorem 2.11 The class $\overline{\beta}$ is convex. Let H_1, H_2 belong to $\overline{\beta}$ and for define $0<\alpha<1$,

$$H_\alpha=\alpha H_1+(1-\alpha)H_2 \tag{2.8}$$

i.e. a convex combination of H_1 and H_2. Then H_α belongs to $\overline{\beta}$.

Proof. Since H_1 and $H_2\in\overline{\beta}$, (2.8) may be written as

$$\begin{aligned} H_\alpha(x_1,\cdots,x_n) &\geq \alpha\Pi_{i=1}^{n}F_i(x_i)+(1-\alpha)\Pi_{i=1}^{n}F_i(x_i) \\ &= \Pi_{i=1}^{n}F_i(x_i), \end{aligned} \tag{2.9}$$

(where F_i's are the marginals of H_i, $i=1,2$) so that H_α is $PLOD$. Moreover,

$$\lim_{\substack{x_j\to\infty \\ j=1,\cdots,n\ ;\ j\neq i}} H_\alpha(x_1,\cdots,x_n)=\alpha F_i(x_i)+(1-\alpha)F_i(x_i)=F_i(x_i), \tag{2.10}$$

It follows from (2.9) and (2.10) that $H_\alpha\in\overline{\beta}$.

In addition, it can be easily be show that the class $\bar{\beta}$ is weakly compact. These two properties of $\bar{\beta}$ indicate the possibility of representing each of the *PLOD* class in terms of their extreme points. This area of research may lead to a wide variety of useful inequalities governing *PLOD* distributions.

Theorem 2.12 Let H_1 and H_2 belong to $\bar{\beta}$ and define H_α as in (2.8). Then

$$H_2 \geq^{PLOD} H_\alpha \geq^{PLOD} H_1.$$

3. Preservation Results

Theorem 3.1 Let $\underline{X} = (X_1, \cdots, X_n)$ and $\underline{Y} = (Y_1, \cdots, Y_n)$ be *PLOD* and let for each i, $i = 1, \cdots, n$, $X_i =^d Y_i$($=^d$ stands for the same distribution). Assume that $\underline{Z} = (Z_1, \cdots, Z_m)$ is independent and $\underline{Z}$ is independent of $\underline{X}$ and $\underline{Y}$ respectively. Then $(\underline{X}, \underline{Z})$ is more *PLOD* than $(\underline{Y}, \underline{Z})$.

Proof. First note that for each component of $(\underline{X}, \underline{Z})$ and $(\underline{Y}, \underline{Z})$ have same pairs of marginal distributions and that $(\underline{X}, \underline{Z})$ and $(\underline{Y}, \underline{Z})$ are *PLOD*. Next,

$$\begin{aligned} P(&X_1 \leq c_1, \cdots, X_n \leq c_n, Z_1 \leq c_{n+1}, \cdots, Z_m \leq c_{n+m}) \\ &= P(X_1 \leq c_1, \cdots, X_n \leq c_n) P(Z_1 \leq c_{n+1}, \cdots, Z_m \leq c_{n+m}) \\ &\geq P(Y_1 \leq c_1, \cdots, Y_n \leq c_n) P(Z_1 \leq c_{n+1}, \cdots, Z_m \leq c_{n+m}) \\ &= P(Y_1 \leq c_1, \cdots, Y_n \leq c_n, Z_1 \leq c_{n+1}, \cdots, Z_m \leq c_{n+m}) \end{aligned}$$

for all $(c_1, \cdots, c_n, c_{n+1}, \cdots, c_{n+m}) \in R^{n+m}$. This completes the proof.

Next, we show the *PLOD* ordering preserves under transformation of univariate increasing function.

Theorem 3.2 Let $\underline{X} = (X_1, \cdots, X_n)$ be more *PLOD* than $\underline{Y} = (Y_1, \cdots, Y_n)$. Assume that $f_i: R \to R$, $i = 1, \cdots, n$, are increasing functions. Then $(f_1(X_1), \cdots, f_n(X_n))$ is more *PLOD* than $(f_1(Y_1), \cdots, f_n(Y_n))$.

Proof. First note that $(f_1(X_1), \cdots, f_n(X_n))$ and $(f_1(Y_1), \cdots, f_n(Y_n))$ are *PLOD* and that for each i, $i = 1, \cdots, n$, $f_i(X_i) =^d f_i(Y_i)$ since $X_i =^d Y_i$. Next,

$$\begin{aligned} P(&f_1(X_1) \leq c_1, \cdots, f_n(X_n) \leq c_n) \\ &= P(X_1 \leq f_1^{-1}(c_1), \cdots, X_n \leq f_n^{-1}(c_n)) \qquad (3.1) \\ &\geq P(Y_1 \leq f_1^{-1}(c_1), \cdots, Y_n \leq f_n^{-1}(c_n)) \\ &= P(f_1(Y_1) \leq c_1, \cdots, f_n(Y_n) \leq c_n) \end{aligned}$$

for all $(c_1, \cdots, c_n) \in R^n$. Thus the proof is complete.

Lemma 3.3 Let $\underline{X}$ be more *PLOD* than $\underline{Y}$. Assume that $\underline{Z}$ is *PLOD* and independent of $\underline{X}$ and $\underline{Y}$. Then $\underline{X}+\underline{Z}$ is more *PLOD* than $\underline{Y}+\underline{Z}$.

Proof. Let H be the joint distribution function of $\underline{Z}$ and $H_i(z_i)$ be the marginal distribution of $\underline{Z}$. Then

$$\begin{aligned} P(X_i+Z_i \le c_i) &= \int_{-\infty}^{\infty} P(X_i \le c_i - z_i \mid Z_i = z_i)\, dH_i(z_i) \\ &= \int_{-\infty}^{\infty} P(X_i \le c_i - z_i)\, dH_i(z_i) \\ &= \int_{-\infty}^{\infty} P(Y_i \le c_i - z_i)\, dH_i(z_i) \\ &= P(Y_i + Z_i \le c_i). \end{aligned} \tag{3.2}$$

Thus for each i, $i=1, \cdots, n$, $X_i+Z_i =^d Y_i + Z_i$. Next,

$$\begin{aligned} P(\bigcap_{i=1}^{n} X_i+Z_i \le c_i) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\bigcap_{i=1}^{n} X_i \le c_i - z_i \mid \underline{Z} = \underline{z})\, dH(z_1, \cdots, z_n) \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\bigcap_{i=1}^{n} X_i \le c_i - z_i)\, dH(z_1, \cdots, z_n) \\ &\ge \Pi_{i=1}^{n}\Big[\int_{-\infty}^{\infty} P(X_i \le c_i - z_i)\, dH_i(z_i)\Big] \\ &= \Pi_{i=1}^{n}[P(X_i+Z_i \le c_i)]. \end{aligned} \tag{3.3}$$

Thus $\underline{X}+\underline{Z}$ is *PLOD*. Similarly, $\underline{Y}+\underline{Z}$ is *PLOD*.

Note that the second equality of (3.3) follows since $\underline{Z}$ is independent of $\underline{X}$ and that the inequality follows since $\underline{X}$ is *PLOD*. As in (3.3) we have

$$\begin{aligned} P(\bigcap_{i=1}^{n} X_i+Z_i \le c_i) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\bigcap_{i=1}^{n} X_i \le c_i - z_i)\, dH(z_i, \cdots, z_n) \\ &\ge \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\bigcap_{i=1}^{n} Y_i \le c_i - z_i)\, dH(z_i, \cdots, z_n) \\ &= P(\bigcap_{i=1}^{n} Y_i + Z_i \le c_i). \end{aligned} \tag{3.4}$$

for all $(c_1, \cdots, c_n) \in R^n$. Thus the proof is complete.

Combining Theorem 3.2 and Lemma 3.3 we have the following result:

Corollary 3.4 Let $\underline{X} = (X_1, \cdots, X_n)$ be more *PLOD* than $\underline{Y} = (Y_1, \cdots, Y_n)$ and let $f_i: R \to R$ be increasing functions. Assume the one dimensional random variable Z is independent of $\underline{X}$ and $\underline{Y}$. Define for each i $U_i = f_i(X_i) + Z$, $V_i = f_i(Y_i) + Z$,Then

$$(U_1, \cdots, U_n) \ge^{PLOD} (V_1, \cdots, V_n).$$

Theorem 3.5 Suppose that $\underline{X}$ is more $PLOD$ than $\underline{U}$ and that $\underline{Y}$ is more $PLOD$ than $\underline{V}$. Further, let $\underline{Y}$ be independent of $\underline{X}$ and $\underline{U}$, respectively and $\underline{U}$ be independent of $\underline{V}$. Then $\underline{X}+\underline{Y}$ is $PLOD$ than $\underline{U}+\underline{V}$.

Proof. Assume that $\underline{X}$ is more $PLOD$ than $\underline{U}$. Specifying $\underline{Z}$ to be $\underline{Y}$ and applying Lemma 3.3 we obtain

$$\underline{X}+\underline{Y}\geq^{PLOD}\underline{U}+\underline{Y} \tag{3.5}$$

Next, by the assumption that $\underline{Y}$ is more $PLOD$ than $\underline{V}$, specifying $\underline{Z}$ to be $\underline{U}$ and applying Lemma 3.3 yield

$$\underline{Y}+\underline{U}\geq^{PLOD}\underline{V}+\underline{U}. \tag{3.6}$$

By combining (3.5) and (3.6) we complete the proof.

Lemma 3.6 Let $\underline{Z}=(Z_1,\cdots,Z_n)$ have independent components, and $\underline{Z}$ be independent of $\underline{X}=(X_1,\cdots,X_n)$ and $\underline{Y}=(Y_1,\cdots,Y_n)$. Let f_i: $R^2\rightarrow R$ be increasing functions. If $\underline{X}$ is more $PLOD$ than $\underline{Y}$ then

$$(f_1(X_1,Z_1),\cdots,f_n(X_n,Z_n))\geq^{PLOD}(f_1(Y_1,Z_1),\cdots,f_n(Y_n,Z_n)) \tag{3.7}$$

Proof. Let $H(z_1,\cdots,z_n)$ be the distribution of $\underline{Z}$ and $H_i(z_i)$ be the marginals. First, by the monotonicity of f_i $\{x_i: f_i(x_i,z_i)\leq c_i\}$ is lower interval and hence

$$\begin{aligned}P(f_i(X_i,Z_i)\leq c_i) &= \int_{-\infty}^{\infty}P(f_i(X_i,z_i)\leq c_i\mid Z_i=z_i)\,dH_i(z_i)\\ &= \int_{-\infty}^{\infty}P(f_i(X_i,z_i)\leq c_i)\,dH_i(z_i)\\ &= \int_{-\infty}^{\infty}P(f_i(Y_i,z_i)\leq c_i)\,dH_i(z_i)\\ &= P(f_i(Y_i,Z_i)\leq c_i).\end{aligned}$$

Thus $f_i(X_i,Z_i)$ and $f_i(Y_i,Z_i)$ have same distributions. Next,

$$\begin{aligned}P(\bigcap_{i=1}^{n}f_i(X_i,Z_i)\leq c_i) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}P(\bigcap_{i=1}^{n}f_i(X_i,z_i)\leq c_i)\,dH(z_i,\cdots,z_n)\\ &\geq \Pi_{i=1}^{n}\Big[\int_{-\infty}^{\infty}P(f_i(X_i,z_i)\leq c_i)\,dH_i(z_i)\Big]\\ &= \Pi_{i=1}^{n}P(f_i(X_i,Z_i)\leq c_i).\end{aligned}$$

Thus $(f_1(X_1,Z_1),\cdots,f_n(X_n,Z_n))$ is $PLOD$. Similarly, $(f_1(Y_1,Z_1),\cdots,f_n(Y_n,Z_n))$ is $PLOD$. Finally, for all $(c_1,\cdots,c_n)\in R^n$.

$$\begin{aligned}P(\bigcap_{i=1}^{n}(f_i(X_i,Z_i)\leq c_i) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}P(\bigcap_{i=1}^{n}(f_i(X_i,z_i)\leq c_i))\,dH(z_1,\cdots,z_n)\\ &\geq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}P(\bigcap_{i=1}^{n}(f_i(Y_i,z_i)\leq c_i)\,dH(z_1,\cdots,z_n)\\ &= P(\bigcap_{i=1}^{n}(f_i(Y_i,Z_i)\leq c_i).\end{aligned}$$

Thus the proof is complete.

Theorem 3.7 Assume

(i) $\underline{X}=(X_1,\cdots,X_n)$ is more $PLOD$ than $\underline{Y}=(Y_1,\cdots,Y_n)$.

(ii) $\underline{U}=(U_1,\cdots,U_n)$ is more $PLOD$ than $\underline{V}=(V_1,\cdots,V_n)$.

(iii) $\underline{U}=(U_1,\cdots,U_n)$ is independent of $\underline{X}$ and $\underline{Y}$.

(iv) $\underline{Y}$ is independent of $\underline{V}$.

(v) $\underline{U}$ and $\underline{Y}$ have independent components, respectively.

Then for increasing functions f_i: $R^2\to R$ $i=1,\cdots,n$,

$$(f_1(X_1,U_1),\cdots,f_n(X_n,U_n)\geq^{PLOD}(f_1(Y_1,V_1),\cdots,f_n(Y_n,V_n)).$$

Proof. Define $f_i(s,t)=f_i'(t,s)$. Then by Lemma 3.6

$$\begin{aligned}(f_1(X_1,U_1),\cdots,f_n(X_n,U_n)) &\geq^{PLOD}(f_1(Y_1,U_1),\cdots,f_n(Y_n,U_n))\\ &=(f_1'(U_1,Y_1),\cdots,f_n'(U_1,Y_n))\\ &\geq^{PLOD}(f_1'(V_1,Y_1),\cdots,f_n'(V_n,Y_n))\\ &=(f_1(Y_1,V_1),\cdots,f_n(Y_n,V_n)).\end{aligned}$$

Thus the proof is complete.

In Theorem 3.8 we will show that the $PLOD$ partial ordering is preserved under limit in distributions:

Theorem 3.8 Assume that H_n and H_n' have same pairs of marginals. Let H_n be more $PLOD$ than H_n' for every n and H_n, H_n' converge weakly to H, H', respectively. Then H is more $PLOD$ than H'.

Proof. Since H_n and H_n' have same pairs of marginals H and H' also have same pairs of marginals. H and H' are $PLOD$ since H_n, H_n' are $PLOD$.

Denote by $C(H)$ and $C(H')$ the sets of continuity points of H and H', respectively. Let $D=C(H)\cap C(H')$. It follows from assumptions that

$$H(c_1,\cdots,c_n)\geq H'(c_1,\cdots,c_n) \text{ for all } (c_1,\cdots,c_n)\in D.$$

Since D is a dense set in R^n

$$H(c_1,\cdots,c_n)\geq H'(c_1,\cdots,c_n) \text{ for all } (c_1,\cdots,c_n)\in R^n.$$

Thus H is more $PLOD$ than H'.

Before we state the next theorem we need a definition and a result of Ahmed et al. (1979).

Definition 3.9 (Barlow and Proschan, 1981) A random variable Y is stochastically increasing(SI) in the random variable X if $E[f(Y)\mid X=x]$ is nondecreasing in x for all real valued, nondecreasing integrable functions f.

Lemma 3.10 (Ahmed et al., 1979) Let (*i*) $\underline{X} = (X_1, \cdots, X_n)$ given λ, be a conditionally *POD*, (*ii*) $X_i \uparrow$ st in λ for $i = 1, \cdots, n$. Then $\underline{X}$ is *POD*.

We may now define the class $\overline{\beta_\lambda}$ by $\overline{\beta_\lambda} = \{H_\lambda : H_\lambda(\infty, \cdots, \infty, x_i, \infty, \cdots, \infty) = F_i(x_i|\lambda)$ for all $i = 1, \cdots, n$, $H_\lambda | \lambda$ is *PLOD*, and X_i's are SI in $\lambda\}$. The following theorem shows that if two elements of $\overline{\beta_\lambda}$ are ordered according to $\geq^{PLOD}$,
then after mixing on λ the resulting element in $\overline{\beta}$ preserve the same order.

Theorem 3.11 Let $\underline{X}|\lambda = (X_1, \cdots, X_n)|\lambda$ and $\underline{Y}|\lambda = (Y_1, \cdots, Y_n)|\lambda$ belong to $\overline{\beta_\lambda}$ and let $\underline{X}|\lambda \geq^{PLOD} \underline{Y}|\lambda$ for all λ. Then, unconditionally, $\underline{X}, \underline{Y}$ belong to $\overline{\beta}$ and $\underline{X} \geq^{PLOD} \underline{Y}$.

Proof. First note that $\underline{X}, \underline{Y}$ belong to $\overline{\beta}$ according to Lemma 3.10. Next,

$$P\left(\bigcap_{i=1}^{n} X_i \leq x_i\right) = E_\lambda\left[P\left(\bigcap_{i=1}^{n} X_i \leq x_i | \lambda\right)\right]$$

$$\geq E_\lambda\left[P\left(\bigcap_{i=1}^{n} Y_i \leq x_i | \lambda\right)\right] = P\left(\bigcap_{i=1}^{n} Y_i \leq y_i\right).$$

Thus the proof is complete. The inequality follows from assumption that $\underline{X}|\lambda \geq^{PLOD} \underline{Y}|\lambda$.

4. An example

Subramanyan(1990) has already studied positive quadrant dependence in three demensions. For completeness we repeat some of the arguments given in that paper and construct some *PLOD* ordering using them. Consider the case where each of X, Y, and Z assumes only two values 1 and 2, say. Let $P_{ijk} = P(X=i, Y=j, Z=k)$, $i=1,2,;$ $j=1,2,;k=1,2.$ The joint probability law of X, Y, and Z is written, for convenience,

$$P = \begin{bmatrix} P_{111} & P_{112} & P_{121} & P_{122} \\ P_{211} & P_{212} & P_{221} & P_{222} \end{bmatrix}.$$

In terms of this new notation, F is *PLOD* if

$$P_{111} \geq p_1 q_1 r_1 \tag{4.1}$$

$$P_{111} + P_{112} \geq p_1 q_1 \tag{4.2}$$

$$P_{111} + P_{121} \geq p_1 r_1 \tag{4.3}$$

$$P_{111} + P_{211} \geq q_1 r_1 \tag{4.4}$$

where $p_1 = P(X=1); q_1 = P(Y=1); r_1 = P(Z=1); p_2 = 1 - p_1; q_2 = 1 - q_1;$ and $r_2 = 1 - r_1$. Let $0 < p_1 < 1$, $0 < q_1 < 1$, and $0 < r_1 < 1$ be three fixed numbers. Let $\overline{\beta}(p_1, q_1, r_1)$ be the collection of all trivariate distributions $P = (P_{ijk})$ with support contained in $\{(i, j, k); i =$

$1, 2, j=1, 2,$ and $k=1, 2\}$ such that F is *PLOD*, and the marginal distributions of X, Y and Z under F are $p_1, 1-p_1; q_1, 1-q_1$, and $r_1, 1-r_1$, respectively.

Any $P=(P_{ijk}) \in \overline{\beta}(p_1, q_1, r_1)$ must satisfy the inequalities (4.1), (4.2), (4.3), and (4.4). Also, due to marginality restrictions, we should have

$$P_{111}+P_{112}+P_{121} \leq p_1, \tag{4.5}$$

$$P_{111}+P_{112}+P_{211} \leq q_1, \tag{4.6}$$

$$P_{111}+P_{121}+P_{211} \leq r_1. \tag{4.7}$$

The following are the natural nonnegativity conditions.

$$P_{112} \geq 0, \tag{4.8}$$

$$P_{121} \geq 0, \tag{4.9}$$

$$P_{211} \geq 0. \tag{4.10}$$

All these inequalities (4.1) to (4.10) involve $P_{111}, P_{112}, P_{121}, P_{211}$ only. If some four numbers $P_{111}, P_{112}, P_{121}, P_{211}$ satisfy the inequalities (4.1) to (4.10) then one could define

$$P_{122}=p_1-(P_{111}+P_{112}+P_{121}), \tag{4.11}$$

$$P_{212}=q_1-(P_{111}+P_{112}+P_{211}), \tag{4.12}$$

$$P_{221}=r_1-(P_{111}+P_{121}+P_{211}), \tag{4.13}$$

$$P_{222}=(1-p_1-q_1-r_1)+P_{111}+(P_{111}+P_{112}+P_{121}+P_{211}). \tag{4.14}$$

The numbers P_{122}, P_{221}, and P_{211} will be nonnegative. If $P_{222} \geq 0$, then

$$P=(P_{ijk}) \in \overline{\beta}(p_1, q_1, r_1).$$

Select 4 inequalities from (4.1) to (4.10) and replace the inequality signs by equality signs. Solve the resultant system of 4 linear equations in 4 unknowns $P_{111}, P_{112}, P_{121}$, and P_{211}. If there is a solution, and this solution satisfies the remaining inequalities, determine P_{122}, P_{212}, P_{221}, and P_{222} as per the equations (4.11), (4.12), (4.13), and (4.14). If $P_{222} \geq 0$, then

$$P=(P_{ijk}) \in \overline{\beta}(p_1, q_1, r_1).$$

Let us define the joint distributions

$$F_U(x, y, z)=F_1(x) \wedge F_2(y) \wedge F_3(z), \tag{4.15}$$

$$F_0(x, y, z)=F_1(x)F_2(y)F_3(z), \text{ for all } x, y \text{ and } z, \tag{4.16}$$

where $F_1(x)=0$, if $x<1$, $=p_1$ if $1 \leq x < 2$, and $=1$ if $x \geq 2$; and $F_2(y)=0$ if $y<1$, $=q_1$ if $1 \leq y < 2$, $=1$and if $y \geq 2$; and $F_3(z)=0$ if $z<1$, $=r_1$ if $1 \leq z < 2$, and $=1$ if $z \geq 2$; and for any two numbers u, v, $u \wedge v$ stands for the minimums of the numbers u and v. $F_U(x, y, z)$ is the upper Frechet bound with marginals F_1, F_2 and F_3.

An explicit computation shows that the corresponding distribution $P_U(=F_U)$ has the follo-wi ng entries

$$P_{111}=p_1 \wedge q_1 \wedge r_1; P_{112}=p_1 \wedge q_1-P_{111}, P_{121}=p_1 \wedge r_1-P_{111};$$

$$P_{211}=q_1\wedge r_1-P_{111};\ P_{221}=r_1-P_{211}-P_{121}-P_{111};$$
$$P_{212}=q_1-P_{112}-P_{211}-P_{111};\ P_{122}=p_1-P_{121}-P_{112}-P_{111};$$
$$P_{222}=1-P_{111}-P_{112}-P_{121}-P_{211}-P_{122}-P_{212}-P_{221}.$$

It can be verfied that the upper Frechet bound is $PLOD$ with same marginals F_1, F_2 and F_3. Similarly we obtain the corresponding distribution $P_0(=F_0)$

$$P_0=\begin{bmatrix} p_1q_1r_1 & p_1q_1r_2 & p_1q_2r_1 & p_1q_2r_2 \\ p_2q_1r_1 & p_2q_1r_2 & p_2q_2r_1 & p_2q_2r_2 \end{bmatrix}$$

and verify that P_0 is $PLOD$ with same marginals F_1, F_2 and F_3. By tedious computations we derive

$$P_U \geq^{PLOD} P \geq^{PLOD} P_0$$

where $P\neq P_0$ and $P\neq P_U$. To purse the above approach we consider the following $PLOD$ table when $p_1=q_1=r_1=\frac{1}{2}$.

< Table >

$$P_0=\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad P_1=\frac{1}{8}\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

$$P_2=\frac{1}{8}\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} \qquad P_3=\frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$P_4=\frac{1}{8}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad P_5=\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$P_6=\frac{1}{8}\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \qquad P_7=\frac{1}{8}\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

$$P_8=\frac{1}{8}\begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & 0 & 0 & \frac{5}{2} \end{bmatrix} \qquad P_9=\frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

Remarks < Table > reveals the following insights :

1. Note that the joint distribution P_4 is the upper Frechet bound, that is, $P_U=P_4\geq^{PLOD} P_i$ for all i, $0\leq i\leq 9$.
2. It can be possible to look for convex combinations of P_9 and some or all of $P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$. For instance, any convex combination $P_\lambda=\lambda P_0+(1-\lambda)P_9$ with $0\leq\lambda<1$ is $PLOD$ and $P_9\geq^{PLOD} P_\lambda \geq^{PLOD} P_0$.
3. It is clear that $P_U\geq^{PLOD} P_i\geq^{PLOD} P_0$, for $i=1,2,3,4,5,6,7,8,9$.

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