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Robustness of Bayes Test on Dependent Sample[1)]

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Abstract

It is well known that the assumption of independence is often not valid for real data. This phenomenon has been observed empirically by many prominent scientists. In this article the sensitivity of dependence on Bayes test of a sharp null hypothesis is considered. The robustness is considered with respect to the significant level and the prior probability on the null hypothesis.

1. Introduction

The sensitivity of statistical procedures to violations of the independence assumption has been studied by several authors mainly in classical analysis. Gastwirth and Rubin(1971) studied the effect of dependence on the level of the one-sample t-test, sign test and Wilcoxon test. Also, the effect of dependence on robust estimators has been studied by Gastwirth and Rubin(1975). Serfling(1968) considered the two-sample Wilcoxon test under strongly mixing processes. Albers(1978) considered the modified t-test which has robustness of validity under m-dependence. Recently, a study of various statistical methods(mainly, estimation) for data with long-range dependence has been performed by Beran(1992a, 1992b) etc. In Lehmann(1986), it is summarized that the level of the well-known t-test is quite sensitive to the assumption of independence even asymptotically in the problem of testing the mean of a normal population.

In the present paper, the effect of dependence on Bayes test of a sharp null hypothesis is considered. Specifically, the data, $x_1, \cdots, x_n$, is taken from a population which has normal distribution with mean θ and variance σ^2. Let ρ_{ij} denote the correlation between X_i and X_j for $i \neq j$. It will be shown that the ordinary Bayes test under the assumption of independence for the unknown mean θ is asymptotically robust if $\lim_{n \to \infty} \frac{1}{n} \sum_{i \neq j} \rho_{ij} < \infty$ and not robust if

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$\lim_{n\to\infty} \frac{1}{n} \sum_{i\neq j} \rho_{ij} = \infty$ and $\lim_{n\to\infty} \frac{1}{n} \sum_{i\neq j} \rho_{ij} = O(n)$. Illustrative examples are also given. The robustness is considered with respect to the significant level and with respect to the prior probability on the null hypothesis.

2. Description of the problem

Let $X_1, \cdots, X_n$ be normally distributed random variables with mean θ and variance σ^2. We wish to test $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ for some specified value of θ_0. Suppose we assign a prior probability of π_0 to the null hypothesis H_0, and use a conjugate prior $N(\theta_0, \tau^2)$ on the alternative. Then the Bayes test is rejecting H_0, if and only if $P(H_0|\boldsymbol{x}) \leq \frac{1}{2}$, where $P(H_0|\boldsymbol{x})$ denotes the posterior probability of H_0, Let ρ_{ij} be the correlation coefficient of X_i and X_j for $i\neq j$. If $\rho_{ij} = 0$ for all i and j with $i \neq j$, i.e., Xs are independent, then, since $\bar{x}$ is a sufficient statistic of θ,

$$P(H_0|\boldsymbol{x}, \rho_{ij}=0) = P(H_0|\bar{x}, \rho_{ij}=0) = \frac{f(\bar{x}\,|\,\theta_0)\pi_0}{m(\bar{x})},$$

where $f(\bar{x}\,|\,\theta_0)$ is the density function of $N(\theta_0, \sigma^2/n)$ and $m(\bar{x})$ is the density function of $N(\theta_0, \sigma^2/n + \tau^2)$ that is the marginal distribution function of $\bar{x}$ for the uncorrelated data. Note that

$$m(\bar{x}) = f(\bar{x}\,|\,\theta_0)\pi_0 + m_g(\bar{x}),$$

where

$$m_g(\bar{x}) = \int_{\theta\neq\theta_0} f(\bar{x}\,|\,\theta) g(\theta)\, d\theta,$$

and $g(\theta)$ is the likelihood function of $N(\theta_0, \tau^2)$. Thus we have (see Berger(1985))

$$P(H_0\,|\,\bar{x}, \rho_{ij}=0) = \left(1 + \frac{1-\pi_0}{\pi_0}\frac{m_g(\bar{x})}{f(\bar{x}\,|\,\theta_0)}\right)^{-1} = \left(1 + \frac{1-\pi_0}{\pi_0}\frac{\exp(\{n\tau(\bar{x}-\theta_0)\}^2/\{2\sigma^2(n\tau^2+\sigma^2)\})}{\sqrt{1+n\tau^2/\sigma^2}}\right)^{-1}. \quad (1)$$

So if the observations are independent each other, the Bayes test is to reject the null hypothesis whenever $(1) \leq 1/2$. Besides, it can be verified that the Bayes test is asymptotically optimal in a classical sense. That is, the level of the above test is

$P_{\theta_0, \rho_{ij}=0}((1) \le 1/2)$, which goes to zero as n tends to infinity. This means if we have enough information of data, Type I error of the Bayes test is small for uncorrelated data. If the observations are not independent, however, the above test may lose its optimality. In this paper, the asymptotic robustness of the above test under dependent data is investigated. Specifically,

$$P_{\theta_0,\ \rho_{ij}\neq 0 \text{ for some } i,j}(P(H_0 \mid \overline{x}, \rho_{ij}=0) \le 1/2) \;=\; P_{\theta_0, \rho_{ij}}((1) \le 1/2)$$

is considered with illustrative examples.

3. Main results

Let us begin with an example.

Example 1. Assume that $\rho_{ij} = \rho > 0$ for all $i \neq j$. Then

$$\frac{1}{n}\sum_{i \neq j} \rho_{ij} = (n-1)\rho.$$

Since $\overline{X}$ is distributed as $\mathcal{N}(\theta_0, \frac{\sigma^2}{n}(1+(n-1)\rho))$ under the null hypothesis H_0,

$$\begin{aligned}
P_{\theta_0, \rho\neq 0}(P(H_0 \mid \overline{X}, \rho=0) \le 1/2) &= P_{\theta_0, \rho\neq 0}((1) \le 1/2)\\
&= P_{\theta_0, \rho\neq 0}\left(\frac{n\overline{X}^2}{2\sigma^2(1+\sigma^2/(n\tau^2))} \ge \ln\left(\frac{\pi_0}{1-\pi_0}\sqrt{\frac{1+n\tau^2}{\sigma^2}}\right)\right)\\
&= P_{\theta_0, \rho\neq 0}\left(\frac{n\overline{X}^2}{\sigma^2(1+(n-1)\rho)} \ge c_n + d_n\right)\\
&= P(\chi_1^2 \ge c_n + d_n),
\end{aligned}$$

where

$$c_n = \frac{2(1+\sigma^2/(n\tau^2))}{1+(n-1)\rho}\ln\frac{\pi_0}{1-\pi_0} \quad \text{and} \quad d_n = \frac{1+\sigma^2/(n\tau^2)}{1+(n-1)\rho}\ln\left(1+\frac{n\tau^2}{\sigma^2}\right).$$

We notice that $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} d_n = 0$ by the L'Hospital law. Then by the Slutsky theorem (Bickel & Doksum(1977)),

$$\begin{aligned}
\lim_{n\to\infty} P(\chi_1^2 \ge c_n + d_n) &\ge P(\chi_1^2 \ge 0)\\
&= 1.
\end{aligned}$$

Thus the test based on $P(H_0 \mid \overline{x}, \rho=0)$ asymptotically derives false conclusion with probability 1 given H_0. For the prior probability $\pi_0=0.5$, Table 1 gives the exact calculation of $P_{\theta_0,\rho\neq 0}(P(H_0|\overline{X},\rho=0)\le 1/2)$ when $\sigma^2/\tau^2=1/2$. It shows that the level of the Bayes test is not robust in small sample and in large sample. Moreover, the level of the Bayes test increases as sample size increases, which is quite a noticeable result.

Next theorem provides us a general phenomenon.

Theorem 1. Suppose that $\lim_{n\to\infty}\frac{1}{n}\sum_{i\neq j}\rho_{ij}=\infty$ and moreover $\frac{1}{n}\sum_{i\neq j}\rho_{ij}=O(n)$. Then $\lim_{n\to\infty}P_{\theta_0,\rho_{ij}\neq 0}(P(H_0|\overline{X},\rho_{ij}=0)\le 1/2)=1$.

Proof : Since $\overline{X}$ is distributed as $\mathcal{N}(\theta_0,\frac{\sigma^2}{n}(1+\frac{1}{n}\sum_{i\neq j}\rho_{ij}))$,

$$\begin{aligned}
P_{\theta_0,\rho_{ij}}(P(H_0|\overline{X},\rho_{ij}=0)\le 1/2) &= P_{\theta_0,\rho_{ij}}((1)\le 1/2)\\
&= P_{\theta_0,\rho_{ij}}\left(\frac{n\overline{X}^2}{\sigma^2(1+1/n\sum_{i\neq j}\rho_{ij})}\ge a_n+b_n\right)\\
&= P_{\theta_0}(\chi_1^2\ge a_n+b_n),
\end{aligned}$$

where

$$a_n=\frac{2(1+\sigma^2/(n\tau^2))}{1+\frac{1}{n}\sum_{i\neq j}\rho_{ij}}\ln\frac{\pi_0}{1-\pi_0}\quad\text{and}\quad b_n=\frac{1+\sigma^2/(n\tau^2)}{1+\frac{1}{n}\sum_{i\neq j}\rho_{ij}}\ln\left(1+\frac{n\tau^2}{\sigma^2}\right).$$

By the assumption, $\lim_{n\to\infty}a_n=0$. Also, since $\frac{1}{n}\sum_{i\neq j}\rho_{ij}=O(n)$,

$$\begin{aligned}
\lim_{n\to\infty}b_n &= \lim_{n\to\infty}\frac{\ln(1+n\tau^2/\sigma^2)}{O(n)}\\
&= \lim_{n\to\infty}\frac{\frac{1}{n}\ln(1+n\tau^2/\sigma^2)}{O(n)/n}\\
&= 0.
\end{aligned}$$

This completes the proof. □

In the assumption of the above theorem, let us consider the prior probability of H_0, π_0, to obtain the significant level α as n goes to infinity. Suppose

$$P_{\theta_0,\rho_{ij}}(P(H_0|\overline{x},\rho_{ij}=0\text{ for all }i,j)\le 1/2)=\alpha.$$

Then

$$a_n+b_n=U_\alpha,$$

where a_n and b_n are defined as proof of Theorem 1, and U_α denotes the αth quartile of

Chisquare distribution with 1 degree of freedom. Solving this equation for π_0, we obtain

$$\pi_0 = \frac{1}{1+\sqrt{1+n\tau^2/\sigma^2}\exp\left(-U_\alpha\frac{1+\frac{1}{n}\sum_{i\neq j}\rho_{ij}}{2(1+\sigma^2/(n\tau^2))}\right)},$$

which tends to 1 as $n\rightarrow\infty$. That is, we need larger prior π_0 to get the significant level α as n goes to infinity. Thus, if $\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i\neq j}\rho_{ij}=\infty$ and $\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i\neq j}\rho_{ij}=O(n)$, the test assuming independence is not asymptotically robust, in fact, it derives false conclusion with probability 1 given H_0.

Let us consider the other case.

Theorem 2. If $\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{1\leq i\neq j\leq n}\rho_{ij}<\infty$ and $\rho_{ij}\neq 0$ for some i,j, then

$$P_{\theta_0,\rho_{ij}}(P(H_0|\bar{x},\rho_{ij}=0 \text{ for all } i,j)\leq 1/2)$$

is asymptotically 0 as n goes to infinity.

Proof : Notice that $P_{\theta_0,\rho_{ij}}(P(H_0|\bar{x},\rho_{ij}=0 \text{ for all } i,j)\leq 1/2) = P_{\theta_0}(\chi_1^2\geq a_n+b_n)$, where a_n and b_n are defined as Theorem 1. Since $\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{1\leq i\neq j\leq n}\rho_{ij}<\infty$, a_n+b_n tends to infinity as n goes to infinity. Then the proof of the theorem is complete. □

Thus the Type I error by using $P(H_0|\bar{x},\rho_{ij}=0$ for all $i,j)$ is asymptotically small if $\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i\neq j}\rho_{ij}<\infty$.

Let us again consider the prior probability of H_0, π_0, to obtain the significant level α as n goes to infinity. Suppose

$$P_{\theta_0,\rho_{ij}}(P(H_0|\bar{x},\rho_{ij}=0 \text{ for all } i,j)\leq 1/2) = \alpha.$$

Solving this equation for π_0, we obtain

$$\pi_0 = \frac{1}{1+\sqrt{1+n\tau^2/\sigma^2}\exp\left(-U_\alpha\frac{1+\frac{1}{n}\sum_{i\neq j}\rho_{ij}}{2(1+\sigma^2/(n\tau^2))}\right)},$$

which tends to 0 as $n\rightarrow\infty$ if $\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i\neq j}\rho_{ij}<\infty$, where U_α denotes the αth quartile of Chisquare distribution with 1 degree of freedom. In other words, we need smaller prior π_0 to have the significant level α as n increases. Hence we now conclude that if $\lim_{n\rightarrow\infty}\frac{1}{n}\sum_{i\neq j}\rho_{ij}<\infty$, it is asymptotically robust to use the test assuming independence. Examples are given below.

Example 2. (Moving-Average Process) Suppose that, for some fixed positive integer $m(< n)$,

$$\rho_{ij} = \begin{cases} \rho_{|i-j|} & \text{if } 1 \le |i-j| \le m \\ 0 & \text{if } |i-j| > m. \end{cases}$$

Then

$$\frac{1}{n} \sum_{i \ne j} \rho_{ij} = 2 \sum_{k=1}^{m} (1 - \frac{k}{n}) \rho_k.$$

In addition, $\frac{1}{n} \sum_{i \ne j} \rho_{ij} = 2 \sum_{k=1}^{m} \rho_k < \infty$. Assume that $1 + 2 \sum_{k=1}^{m} \rho_k > 0$. Then by the theorem above, for given $\rho_k \ne 0$ for some k, $P_{\theta_0, \rho_k}(P(H_0 | \overline{x}, \rho_{ij} = 0 \text{ for all } k) \le 1/2)$ is asymptotically 0 as n goes to infinity.

Example 3. (First-Order Autoregressive Process) Let us consider the following correlation matrix:

$$\begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}$$

where $|\rho| < 1$. In this case

$$\frac{1}{n} \sum_{i \ne j} \rho_{ij} = 2 \sum_{k=1}^{n-1} (1 - \frac{k}{n}) \rho^k$$

which converges to $2\rho/(1-\rho) < \infty$. Thus the condition of Theorem 2 is satisfied. Table 2. gives the exact calculation of $P_{\theta_0, \rho \ne 0}(P(H_0 | \overline{X}, \rho = 0) \le 1/2)$ when the prior probability $\pi_0 = 0.5$ and $\sigma^2/\tau^2 = 1/2$.

<Table 1> The level of the Bayes test in Example 1 when the prior probability $\pi_0 = 0.5$ and $\sigma^2/\tau^2 = 1/2$.

ρ \ n	10	20	30	50	100
0.05	0.1376	0.1624	0.1915	0.2451	0.3439
0.1	0.1946	0.2519	0.3007	0.3741	0.4844
0.5	0.4458	0.5471	0.6036	0.6609	0.7453

<Table 2> The level of the Bayes test in Example 3 when the prior probability $\pi_0 = 0.5$ and $\sigma^2/\tau^2 = 1/2$.

ρ \ n	10	20	30	50	100
0.05	0.0874	0.0628	0.0514	0.0398	0.0280
0.1	0.1023	0.0761	0.0635	0.0504	0.0366
0.5	0.2675	0.2436	0.2273	0.2064	0.1796

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