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Multivariate EWMA Charts for Simultaneously Monitoring both Means and Variances[1)]

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Abstract

Multivariate control statistics to simultaneously monitor both means and variances for several quality variables under multivariate normal process are proposed. Performances of the proposed multivariate charts are evaluated in terms of average run length(ARL). Multivariate Shewhart chart is also proposed to compare the performances of multivariate exponentially weighted moving average(EWMA) charts. A numerical comparison shows that multivariate EWMA charts are more efficient than multivariate Shewhart chart for small and moderate shifts and multivariate EWMA scheme based on accumulate-combine approach is more efficient than corresponding multivariate EWMA chart based on combine-accumulate approach.

1. Introduction

Control charts are widely used to display sample data from a process for purposes of determining whether a process is in-control, for bringing an out-of-control process into control, and for monitoring a procsess to make sure that it stays in-control. When the chart signals that an assignable cause is present, then a rectifying action is taken to remove the assignable cause and bring the process back into control. The ability of a control chart to detect process changes is determined by the length of time required for the chart to signal.

The quality of a product is often characterized by joint levels of several correlated variables rather than a single characteristic. When the quality characteristics are correlated, one can obtain better sensitivity by using multivariate control chart than separate control charts for each of the quality parameters. The control procedure using separate charts for each individual parameter may not be optimal for detecting simultaneous changes in means and variances.

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The original work on multivariate control chart was introduced by Hotelling(1947) and multivariate approach for control chart has become increasingly popular in recent years. Alt(1982) reviewed much of the study on multivariate control charts. Ghare and Togersen(1968) presented multivariate Shewhart control chart based on Hotelling's T^2 statistic to control means of two variables simultaneously.

The EWMA chart, first developed by Roberts(1959), has recently received a great deal of attention in quality control literatures as a process monitoring tool, primarily to detect a small or moderate shift of the process. In EWMA chart, the most recent observations are assigned more weights and the older observations are assigned less weights. Like the Shewhart chart, the EWMA control chart is easy to implement and interpret.

Most of the studies on multivariate control charts have been concentrated for monitoring mean vector of multivariate normal process. In this paper, we present a single multivariate control scheme to simultaneously monitor both the means and variances of the multivariate normal process.

2. Multivariate Shewhart Chart

Assume that the process of interest has $p(p\geq 2)$ correlated quality characteristics represented by the random vector $\underline{X}=(X_1,X_2,\cdots,X_p)'$ and we obtain a sequence of random vectors $\underline{X}_1, \underline{X}_2,\cdots,$ where $\underline{X}_i=(\underline{X}'_{i1},\cdots,\underline{X}'_{in})'$ is a sample of observations at the sampling time i and $\underline{X}_{ij}=(X_{ij1},\cdots,X_{ijp})'$. It will be also assumed that the sequential observation vectors between and within samples are independent and identically distributed and the process quality variables have a multivariate normal distribution. Hence, the distribution of $\underline{X}$ is indexed by a set of parameters $\underline{\theta}=(\underline{\mu},\Sigma)$ where $\underline{\mu}$ is the mean vector and Σ is the covariance matrix of $\underline{X}$. Let $\underline{\theta}_o=(\underline{\mu}_0,\Sigma_0)$ be the known target values for $\underline{\theta}$ of p related quality characteristics. For simplicity, we assume that $\underline{\mu}_0 = \underline{0}$,' all diagonal and off-diagonal elements of Σ_0 are 1 and 0.3, respectively.

To control both μ and σ^2 of a single quality characteristic, Reynolds and Ghosh(1981) proposed an obvious method to use the statistic $U_i + V_i$ and the chart signals if

$$U_i + V_i = \sum_{j=1}^{n}(X_{ij}-\mu_0)^2/\sigma_0^2 \geq \chi^2_{1-\alpha}(n),$$

where $U_i = n(\overline{X}_i-\mu_0)^2/\sigma_0^2$ and $V_i=\sum_{j=1}^{n}(X_{ij}-\overline{X}_i)^2/\sigma_0^2$. Extending the sample statistic $U_i + V_i$ to multivariate case, we obtain the sample statistic D_i for multivariate control chart

$$D_i = \sum_{j=1}^{n}(\underline{X}_{ij} - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{X}_{ij} - \underline{\mu}_0). \tag{2.1}$$

If the process is in-control, the statistic D_i has a chi-squared distribution with np degrees of freedom. When the process has changed to $\underline{\mu}$ from the target mean vector $\underline{\mu}_0$, D_i has a noncentral chi-squared distribution with np degrees of freedom with noncentrality parameter $\tau^2 = n(\underline{\mu} - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{\mu} - \underline{\mu}_0)$. The percentage point of D_i can be obtained from the chi-square distribution when the process are in-control or target mean vector has changed.

Shewhart chart, one of the most widely used control chart, has a good ability to detect large changes in monitored parameter quickly and is easy to implement the process. A multivariate Shewhart control chart based on D_i in (2.1) signals whenever

$$D_i \geq \chi^2_{1-\alpha}(np). \tag{2.2}$$

Result 2.1. Let $\underline{X}_{ij} = (X_{ij1}, X_{ij2}, \cdots, X_{ijp})'$ be distributed according to $N_p(\underline{\mu}_0, \Sigma_0)$ and $\underline{X}_{ij}$'s be independent. If the process parameters of the distribution are changed as $N(\underline{\mu}, c\Sigma_0)$ where c is a constant, then

$$ARL = \frac{1}{1 - F(h/c)},$$

where h is the control limit of the chart in (2.2) and $F(\cdot)$ is a chi-squared distribution function with np degrees of freedom and noncentrality parameter $n(\underline{\mu} - \underline{\mu}_0)'\Sigma_0^{-1}(\underline{\mu} - \underline{\mu}_0)/c$.

3. Multivariate EWMA Chart

There are two basic ways to use the past sample information in multivariate quality control chart. The first, which is called combine-accumulate approach, combines the multivariate data into a univariate statistic and then accumulate over past samples. The second, which is called accumulate-combine approach, accumulates past sample information for each process parameter and then combine the multivariate accumulations into a univariate statistic.

3.1 Combine-Accumulate Approach

A multivariate EWMA chart for monitoring both means and variances based on the control statistic $D_i (i = 1, 2, \cdots)$ in (2.1) is given by

$$Y_{D,i} = (1-\lambda) Y_{D,i-1} + \lambda D_i, \tag{3.1}$$

where $Y_{D,0} = \omega \, (\omega \geq 0)$ and $\lambda \; (0 < \lambda \leq 1)$ is a smoothing constant. This chart signals whenever $Y_{D,i} \geq h$. When the smoothing constant λ is 1, this chart changes to multivariate Shewhart chart.

When the parameters are on-target or mean vector $\underline{\mu}$ has shifted, the performances of this chart can be evaluated by the Markov chain approach. When the process changes on variances, or means and variances, the performances of this chart can be evaluated by simulations. The parameter h can be obtained to satisfy a specified ARL by Markov chain approach.

3.2 Accumulate-Combine Approach

In this section, we propose a procedure which uses two separate EWMA charts, based on accumulate-combine approach, for means and for variances. This proposed scheme signals if either of these two charts signals.

Chart for Means

Lowry et al.(1992) proposed a MEWMA chart for the mean vector of quality characteristics with accumulate-combine technique. He asserted that MEWMA chart for mean vector is a more straightforward generalization of the corresponding univariate procedure than the multivariate CUSUM statistics by Crosier(1988) and Pignatiello and Runger(1990).

MEWMA chart for the mean vector, proposed by Lowry et al.(1992), is a multivariate extension of univariate EWMA chart. The vectors of EWMA's are defined as

$$\underline{Y}_i = (I-\Lambda)\,\underline{Y}_{i-1} + \Lambda\,\overline{\underline{X}}_i \tag{3.2}$$

$i=1,2,\cdots$ where $\overline{\underline{X}}_i$ is the sample mean vector of p quality variables at sampling time i, $\underline{Y}_0 = \underline{\mu}_0$ and $\Lambda = diag(\lambda_1,\cdots,\lambda_p)$, $0 < \lambda_j \le 1$ $(j=1,2,\cdots,p)$. Equation (3.2) can be rewritten by repeated substitution as

$$\underline{Y}_i = \sum_{k=1}^{i} \Lambda(I-\Lambda)^{i-k}\,\overline{\underline{X}}_k + (I-\Lambda)^i\,\underline{\mu}_0 \ . \tag{3.3}$$

Then the dispersion matrix of $\underline{Y}_i$ can be obtained as

$$\begin{aligned} \Sigma_{\underline{Y}_i} &= \sum_{k=1}^{i} Var(\Lambda(I-\Lambda)^{i-k}\,\overline{\underline{X}}_k) \\ &= \sum_{k=1}^{i} [\Lambda(I-\Lambda)^{i-k}\Sigma(I-\Lambda)^{i-k}\Lambda]/n, \end{aligned} \tag{3.4}$$

where Σ is a known covariance matrix. This MEWMA chart for means signals as soon as

$$T^2_{1,i} = (\underline{Y}_i - \underline{\mu}_0)'\,\Sigma_{\underline{Y}_i}^{-1}(\underline{Y}_i - \underline{\mu}_0) > h_1,$$

where $h_1\,(>0)$ is chosen to achieve a specified in-control ARL. Under the assumption that $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda$, the MEWMA vectors can be written as

$$\underline{Y}_i = (1-\lambda)\,\underline{Y}_{i-1} + \lambda\,\overline{\underline{X}}_i \tag{3.5}$$

$i=1,2,\cdots$, and the dispersion matrix of $\underline{Y}_i$ is given by

$$\Sigma_{\underline{Y}_i}=\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]\frac{\Sigma}{n}. \tag{3.6}$$

Lowry et al.(1992) also showed that the distribution of $T^2_{1,i}$ depends on $\underline{\mu}$ and Σ only through the noncentrality parameter τ as

$$\tau=\left[n(\underline{\mu}-\underline{\mu}_0)'\Sigma^{-1}(\underline{\mu}-\underline{\mu}_0)\right]^{1/2}$$

and smaller values of λ are more effective in detecting the smaller shifts in the mean vectors.

Chart for Variances

In the univariate case, the EWMA chart for σ^2 can be constructed by using the chart statistic

$$Y_i=(1-\lambda)Y_{i-1}+\lambda\sum_{j=1}^{n}\left(\frac{X_{ij}-\mu_0}{\sigma_0}\right)^2 \tag{3.7}$$

$i=1,2,\cdots$ where $\mu_0,\ \sigma_0^2$ are known parameters and $0<\lambda\le 1$. By repeated substitution in (3.7), it can be shown that

$$Y_i=(1-\lambda)^iY_0+\sum_{k=1}^{i}\lambda(1-\lambda)^{i-k}\sum_{j=1}^{n}\left(\frac{X_{kj}-\mu_0}{\sigma_0}\right)^2. \tag{3.8}$$

Multivariate EWMA chart for variances with accumulate-combine approach can be constructed by forming multivariate extension of the univariate EWMA chart.

In multivariate case, we define the vectors of EWMA's

$$\underline{Y}_i=\begin{bmatrix}Y_{i1}\\Y_{i2}\\\vdots\\Y_{ip}\end{bmatrix}=\begin{bmatrix}(1-\lambda_1)^iY_{10}+\sum_{k=1}^{i}\lambda_1(1-\lambda_1)^{i-k}\left[\sum_{j=1}^{n}\left(\frac{X_{kj1}-\mu_{01}}{\sigma_{01}}\right)^2-n\right]\\(1-\lambda_2)^iY_{20}+\sum_{k=1}^{i}\lambda_2(1-\lambda_2)^{i-k}\left[\sum_{j=1}^{n}\left(\frac{X_{kj2}-\mu_{02}}{\sigma_{02}}\right)^2-n\right]\\\vdots\\(1-\lambda_p)^iY_{p0}+\sum_{k=1}^{i}\lambda_p(1-\lambda_p)^{i-k}\left[\sum_{j=1}^{n}\left(\frac{X_{kjp}-\mu_{0p}}{\sigma_{0p}}\right)^2-n\right]\end{bmatrix}$$

where $0<\lambda_l\le 1\ (l=1,2,\cdots,p)$ and $i=1,2,\cdots$.

Then the multivariate EWMA vectors can be expressed as

$$\underline{Y}_i=(I-\Lambda)\ \underline{Y}_{i-1}+\Lambda\ \underline{Z}_i, \tag{3.9}$$

where $\underline{Y}_0=\underline{0}$, $\underline{Z}_i=(Z_{i1},Z_{i2},\cdots,Z_{ip})'$ and $Z_{il}=\sum_{j=1}^{n}\left(\frac{X_{ijl}-\mu_{0l}}{\sigma_{0l}}\right)^2-n\ (l=1,2,\cdots,p)$. By repeated substitution in (3.9), $\underline{Y}_i$ can be rewritten as

$$\underline{Y}_i=\sum_{k=1}^{i}\Lambda(I-\Lambda)^{i-k}\ \underline{Z}_k+(I-\Lambda)^i\ \underline{Y}_0 \tag{3.10}$$

$i=1,2,\cdots$ where $\Lambda=diag(\lambda_1,\lambda_2,\cdots,\lambda_p)$, $0<\lambda_j\leq 1\ (j=1,2,\cdots,p)$.

Then we can obtain the dispersion matrix of $\underline{Y}_i$ as

$$\Sigma_{\underline{Y}_i}=\sum_{k=1}^{i}\Lambda(I-\Lambda)^{i-k}\Sigma_{\underline{Z}}(I-\Lambda)^{i-k}\Lambda, \tag{3.11}$$

where $\Sigma_{\underline{Z}}$, dispersion matrix of $\underline{Z}$, is given in Theorem 3.1.

Theorem 3.1. When the process is in-control, the dispersion matrix of $\underline{Y}_i$ is given by

$$\Sigma_{\underline{Y}_i}=\sum_{k=1}^{i}\Lambda(I-\Lambda)^{i-k}\Sigma_{\underline{Z}}(I-\Lambda)^{i-k}\Lambda$$

and

$$\Sigma_Z=2nR^{(2)},$$

where $R^{(2)}$ is used to denote the matrix whose (i,j)th component is the $2nd$ power of the (i,j)th component of R which is the correlation matrix of $\underline{X}=(X_1,X_2,\dots,X_p)$.

Proof. It is easy to show that

$$\begin{aligned}\Sigma_{\underline{Y}_i}&= Var[(I-\Lambda)Y_{i-1}+\Lambda\,\underline{Z}_i]\\ &=\sum_{k=1}^{i}\Lambda(I-\Lambda)^{i-k}\Sigma_{\underline{Z}_i}(I-\Lambda)^{i-k}\Lambda.\end{aligned}$$

To show the form of $\Sigma_{\underline{Z}}$, we obtain the mean vector and variance-covariance matrix of $\underline{Z}_i$ when the process is in-control and $\underline{Y}_0=\underline{0}$. Since a random sample $(X_{i1l},X_{i2l},\cdots,X_{inl})'$ at the sampling point i follows a multivariate normal distribution, the statistic $\sum_{j=1}^{n}(\frac{X_{ijl}-\mu_{ol}}{\sigma_{ol}})^2 = Z_{il}+n$ has a chi-squared distribution with n degrees of freedom. Thus $E(Z_{il})=0$ and $Var(Z_{il})=2n$ for $l=1,2,\cdots,t$ and $i=1,2,\cdots$. Now, for $l\neq s$, we can derive as

$$\begin{aligned}Cov[Z_{il},Z_{is}]&= Cov[Z_{il}+n,Z_{is}+n]\\ &= Cov\left[\sum_{j=1}^{n}(\frac{X_{ijl}-\mu_{0l}}{\sigma_{ol}})^2,\ \sum_{j=1}^{n}(\frac{X_{ijs}-\mu_{0s}}{\sigma_{os}})^2\right]\\ &= nCov\left[(\frac{X_{ijl}-\mu_{0l}}{\sigma_{0l}})^2,(\frac{X_{ijs}-\mu_{0s}}{\sigma_{0s}})^2\right].\end{aligned}$$

If we let $U=\frac{X_{ijl}-\mu_{0l}}{\sigma_{0l}}$ and $V=\frac{X_{ijs}-\mu_{0s}}{\sigma_{0s}}$, then the random variables U and V have a bivariate normal distribution with $N_2(0,0,1,1,\rho_{u,v})$. Using the moment generating function of bivariate normal distribution, $E(U^2V^2)=1+2\rho^2_{u,v}$ and then we can easily obtain that

$$\begin{aligned} Cov(Z_{il}, Z_{is}) &= nCov(U^2, V^2) \\ &= n[E(U^2V^2) - E(U^2)E(V^2)] \\ &= 2n\rho^2_{u,v}. \end{aligned}$$

Thus, we found that the diagonal elements and corresponding off-diagonal elements of Σ_{Z_i} is $2n$ and $2n\rho^2_{i,j}$ from the above results. Therefore,

$$E(\underline{Z}_i) = \underline{0} \quad \text{and} \quad \Sigma_{Z_i} = 2nR^{(2)}. \qquad \square$$

This multivariate chart for variances signals whenever

$$T^2_{2,i} = \underline{Y}_i' \, \Sigma_{\underline{Y}_i}^{-1} \, \underline{Y}_i > h_2,$$

where the parameter h_2 can be obtained to achieve a specified in-control ARL. Under the assumption $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda$, then we can simplify the variance-covariance matrix of $\underline{Y}_t$ as

$$\Sigma_{\underline{Y}_i} = \frac{\lambda[1-(1-\lambda)^{2i}]}{2-\lambda} \cdot \Sigma_{\underline{Z}}. \tag{3.12}$$

The multivariate EWMA scheme based on accumulate-combine approach for simultaneously monitoring both means and variances signals whenever $T^2_{1,i} > h_1$ or $T^2_{2,i} > h_2$. The overall false alarm probability of the chart based on $(T_{1,i}^2, T_{2,i}^2)$ is $1-(1-\alpha_1)(1-\alpha_2)$ where the

<Table 1> ARL for multivariate chart for means and variances ($p=2$)

types of shift	Shewhart	EWMA					
		C-A Approach			A-C Approach		
		$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$	$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$
no shift	200.0	200.0	200.0	200.0	200.0	200.1	200.0
M_1	161.3	137.0	140.4	144.8	30.3	43.2	56.5
M_2	20.6	20.0	14.5	13.3	2.9	3.1	3.3
M_3	3.1	8.2	5.1	4.0	1.3	1.4	1.4
V_1	82.6	60.0	57.5	61.1	46.7	58.1	66.5
V_2	11.9	17.2	11.8	10.3	5.2	6.0	6.8
V_3	4.1	10.1	6.4	5.2	2.4	2.7	2.9
(M_1, V_1)	69.1	50.4	47.1	49.0	20.4	26.1	31.9
(M_2, V_2)	4.7	10.8	6.8	5.5	2.3	2.5	2.6
(M_3, V_3)	1.6	5.3	3.3	2.6	1.3	1.3	1.3

signal probabilities of the chart for means and variances are α_1 and α_2 , respectively. Since it is difficult to obtain the joint distribution of $T^2_{1,i}$ and $T^2_{2,i}$, we obtain the parameters h_1, h_2 and performances of this scheme by simulation.

4. Computational Results and Concluding Remarks

In this paper, we propose a single multivariate EWMA scheme for simultaneously controlling the shift of means and variances of the multivariate normal process. In order to evaluate the performances and compare the proposed multivariate charts, some kinds of standards for comparison are necessary. In our computation, the ARL of the proposed multivariate charts when the process is in-control were fixed to be 200 and the sample size for each characteristic was five for p=2 and p=4. The types of shifts in the parameters when the process is out-of-control for comparison can be stated as follows :

1) Shift in Means : M_i - one mean shifted with $\tau^2 = [(3i-2)/2]^2$.
2) Shift in Variances : V_i - one variance is increased to $[1.0+(3i-2)/10]^2$.
3) Shift in Means and Variances : (M_i, V_i) for $i=1,2,3$.

After the smoothing constants of the proposed multivarite EWMA schemes have been determined, the parameters h were calculated by Markov chains with the number of transient states $r = 100$ or simulation with 10,000 iterations. When the process is in-control or only

<Table 2> ARL for multivariate charts for means and variances ($p=4$)

types of shift	Shewhart	EWMA					
		C-A Approach			A-C Approach		
		$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$	$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$
no shift	200.0	200.0	200.0	200.0	200.3	200.1	200.1
M_1	173.4	153.2	155.8	159.4	38.6	57.5	76.1
M_2	34.3	28.9	21.7	20.8	3.4	3.7	4.0
M_3	5.1	12.6	7.6	6.0	1.5	1.6	1.6
V_1	108.3	80.0	78.4	82.8	59.8	77.8	89.3
V_2	19.0	24.4	17.0	15.2	6.2	7.3	8.4
V_3	6.0	14.8	9.2	7.3	2.7	3.0	3.2
(M_1, V_1)	94.4	69.1	66.2	69.5	26.3	35.5	44.1
(M_2, V_2)	9.3	15.9	10.0	8.1	2.6	2.8	3.0
(M_3, V_3)	2.0	8.2	4.9	3.7	1.4	1.4	1.5

mean vector of the quality variables has changed, the ARL of the multivariate Shewhart chart can be obtained from the chi-square distribution. The numerical results were stated in Tables 1-2.

When small or moderate changes in the process have occurred, EWMA charts are more efficient than Shewhart chart in our computation. We recommend a multivariate EWMA scheme by accumulate-combine approach for simultaneously monitoring both means and variances and Tables 1-2 show that a multivariate EWMA scheme by accumulate-combine feature is more efficient for all types of shift than a multivariate EWMA chart based on the combine-accumulate feature. In our computation, smaller values of smoothing constant are more effective in detecting shifts for multivariate EWMA scheme with accumulate-combine technique.

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