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A Wald Test for a Unit Root Based on the Symmetric Estimator

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Abstract

For an AR(1) model with intercept $y_t = \mu + \rho y_{t-1} + e_t$, a test for random walk hypothesis $H_0: (\mu, \rho) = (0, 1)$ is proposed, which is based on the symmetric estimator. In the vicinity of the null, the test is shown to be more powerful than the test of Dickey and Fuller(1981) based on the ordinary least squares estimator.

1. Introduction

Since the seminal work of Dickey and Fuller(1979), testing for a unit root attracted much attention both from statisticians and economists. Nelson and Plosser(1982) and many others showed that many macro economic variables such as GNP, employment, consumer prices, wages and others are shown to be better modeled by a nonstationary unit root autoregression than a deterministic trend with stationary error.

Owning to the wide applicability of unit root test procedures, a broad class of literature is growing up in various aspects of the statistical method. One direction is to widen the applicability of unit root tests. Many authors developed test procedures which work for autoregessive moving average errors and mixing errors. Also various multivariate extensions were made in connection with cointegration.

Another direction is to improve powers of tests for unit roots. Among those are the works: the symmetric test and the weighted symmetric test (Park and Fuller, 1995), the unconditional test (Gonzalez-Farias, 1992), the invariant test (Elliott *et al.*, 1996). Pantula *et al.*(1994) conducted a Monte-Carlo comparison of those tests including also the classical conditional tests of Dickey and Fuller(1979). They concluded that the weighted symmetric test has the best power. Shin and So(1997) developed symmetric semiparametric tests and showed that the symmetric tests have better power performance than the semiparametric tests of Phillips(1987) and Phillips and Perron(1988).

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All the above procedures test for one parameter, the unit root. Usually the main object of unit root analysis is to show that the economic series under study is a random walk. In autoregressive fittings, in addition to the lags of observations, an intercept term is usually included. Then the hypothesis of random walk is the joint hypothesis of unit root and zero intercept, for which we have the likelihood ratio tests of Dickey and Fuller(1981) based on the ordinary least squares estimator and the Wald tests based on the instrument variable estimator (Hall, 1992). We now observe that the tests for the joint hypothesis can also be improved by considering the better symmetric estimator. Our test is more powerful than the likelihood ratio test of Dickey and Fuller(1981) for the parameter close to null and can be used as a supplementary test to the test of Dickey and Fuller(1981).

In Section 2, we develop a test for the joint hypothesis which is based on the symmetric estimator and the Wald principle. In Section 3, a Monte-Carlo experiment is conducted to study finite sample performance of the proposed test. In Section 4, a concluding remark is given.

2. Wald test statistics

Consider a time series model

$$y_t = \mu + \rho y_{t-1} + e_t, \quad t = 2, \dots, n, \tag{2.1}$$

where $\{y_t\}_{t=1}^{n}$ is a set of observations and e_t is an independent normal $N(0, \sigma^2)$ error sequence. Since mean adjusted fitting is the most widely used for testing unit root purpose, we consider model (2.1) with an intercept.

Of great interest in the economic and statistic literature is to determine whether y_t is a random walk or not. When the true value of (μ, ρ) is (0,1), the true behavior of y_t is a random walk. Various tests for $\rho = 1$ were developed by numerous people such as Dickey and Fuller(1979), Phillips(1987), Phillips and Perron(1988), to name a few. Also tests for combined hypothesis (μ, ρ) = (0,1) were developed by Dickey and Fuller(1981) using the likelihood ratio principle. All the above tests are essentially based on the conditional ordinary least squares estimator

$$\bar{\rho} = \sum_{t=2}^{n} (y_t - \bar{y}_{(0)})(y_{t-1} - \bar{y}_{(-1)}) / \sum_{t=2}^{n} (y_{t-1} - \bar{y}_{(-1)})^2 \text{ and } \bar{\mu} = \bar{y}_{(0)} - \bar{\rho}\, \bar{y}_{(-1)}, \tag{2.2}$$

where $\bar{y}_{(i)} = (n-1)^{-1} \sum_{t=2}^{n} y_{t+i}$ for $i = -1, 0$.

Shin and So(1997) and Park and Fuller(1995) noticed that the AR(1) process $y_t = \mu + \rho y_{t-1} + e_t$ has a forward regression representation

$$y_t = \mu + \rho y_{t-1} + e_t, \quad t=2, \cdots, n \tag{2.3}$$

and also a backward regression representation

$$y_{t-1} = \mu + \rho y_t + b_t, \quad t=2, \cdots, n \tag{2.4}$$

for some error sequence b_t. Under the null $(\mu, \rho) = (0, 1)$, $b_t = -e_t$. In estimating ρ, if we combine both the regression equations (2.3) and (2.4), we get a better estimator of ρ which has less variation than the ordinary least squares estimator when ρ is close to one. This fact gives improved power for the unit root tests based on the symmetric estimator. See Shin and So(1997), Park and Fuller(1995) and Pantula, Gonzalez-Farias and Fuller(1994).

We now consider symmetric estimation in constructing a test for the joint hypothesis $H_0: (\mu, \rho) = (0, 1)$. Let $\theta = (\mu, \rho)'$ be a vector of the parameters. Combining (2.3) and (2.4), we get a regression model

$$Y = X\theta + U, \tag{2.5}$$

where $Y = (y_2, \cdots, y_n, y_1, \cdots, y_{n-1})'$, $X = \{(1, \cdots, 1)', (y_1, \cdots, y_{n-1}, y_2, \cdots, y_n)'\}$, and $U = (e_2, \cdots, e_n, b_2, \cdots, b_n)'$.

Our symmetric estimator of θ is

$$\hat{\theta} = (X'X)^{-1}(X'Y), \tag{2.6}$$

which is simplified as

$$\hat{\rho} = 2\sum_{t=2}^{n}(y_{t-1} - \bar{y})(y_t - \bar{y}) / [\sum_{t=2}^{n}(y_{t-1} - \bar{y})^2 + \sum_{t=2}^{n}(y_t - \bar{y})^2] \tag{2.7}$$

and

$$\hat{\mu} = (1 - \hat{\rho})\bar{y}, \tag{2.8}$$

where $\bar{y} = (2n-2)^{-1}\sum_{t=2}^{n}(y_t + y_{t-1}) = (\bar{y}_{(0)} + \bar{y}_{(-1)})/2$. Observe that, if $\rho = 1$, then $\hat{\mu} = (1 - \hat{\rho})\bar{y} \cong 0$ and $\hat{\mu}$ estimates 0 regardless of the true value of μ. Also note that the term in the first bracket of

$$\hat{\mu} = \{\bar{y}_{(0)} - \hat{\rho}\,\bar{y}_{(-1)}\} - \{2^{-1}(1 + \hat{\rho})(y_n - y_1)/(n-1)\} \tag{2.9}$$

consistently estimates μ. However, the term in the latter bracket of (2.9) induces a bias when μ is not zero and ρ is close to one. Therefore, instead of $\hat{\mu}$, we use a bias-adjusted estimator

$$\tilde{\mu} = \hat{\mu} + \{2^{-1}(1 + \hat{\rho})(y_n - y_1)/(n-1)\}. \tag{2.10}$$

Now, our test for $(\mu, \rho) = (0,1)$ is based on $\tilde{\theta}=(\tilde{\mu}, \hat{\rho})'$. The Wald principle gives us the statistic

$$\hat{\Phi}_s=(\tilde{\theta}-\theta_0)'(X'X)(\tilde{\theta}-\theta_0)/(2\bar{\sigma}^2), \tag{2.11}$$

where $\theta_0=(0,1)'$ is the vector of the parameters under H_0 and

$$\bar{\sigma}^2=\sum_{t=2}^{n}(y_t-\tilde{\mu}-\hat{\rho}y_{t-1})^2/(n-3) \tag{2.12}$$

is an estimator of σ^2 based on the ordinary least squares estimator. Limiting distribution of the test statistic is given in the following Theorem 1.

Theorem 1. Assume $\mu=0$ and $\rho=1$. Then the limiting distribution of $\hat{\Phi}_s$ is $\eta' H\eta$, where

$$\eta=\{2^{-1}(G-V^2)^{-1}V+T, -2^{-1}(G-V^2)^{-1}\}',$$

$$H=\begin{bmatrix} 1 & V \\ V & G \end{bmatrix}, \quad G=\int_0^1 W^2(r)dr, \quad T=W(1), \quad V=\int_0^1 W(r)dr,$$

and $W(r)$ is a standard Brownian motion on [0,1].

Proof. By the invariance principle(Billingsley, 1968, Theorem 16.1), $n^{-1/2}y_{[nr]} \xrightarrow{d} \sigma W(r)$, where $\xrightarrow{d}$ denotes convergence in distribution. Hence, by the continuous mapping theorem (Billingsley, 1968, Theorem 5.1),

$$(n^{-1/2}y_n, n^{-3/2}\sum y_t, n^{-2}\sum y_t^2) \xrightarrow{d} (\sigma T, \sigma V, \sigma^2 G). \tag{2.13}$$

Also

$$n(\hat{\rho}-1)=-\sum e_t^2/[\sum_{t=2}^{n}(y_{t-1}-\bar{y})^2+\sum_{t=2}^{n}(y_t-\bar{y})^2] \xrightarrow{d} -2^{-1}(G-V^2)^{-1} \tag{2.14}$$

and

$$n^{1/2}\tilde{\mu}=(1-\hat{\rho})\bar{y}+\{2^{-1}(1+\hat{\rho})(y_n-y_1)/(n-1)\} \xrightarrow{d} 2^{-1}(G-V^2)^{-1}V\sigma+T\sigma. \tag{2.15}$$

Therefore, jointly, $D_n^{-1}X'X D_n^{-1} \xrightarrow{d} 2H$ and $D_n(\tilde{\theta}-\theta_0) \xrightarrow{d} \sigma\eta$, where $D_n=\text{diag}(n^{1/2}, n\sigma)$ and the result follows.

The limiting distribution in Theorem 1 is different from that of Dickey and Fuller(1981, p.1061). Therefore, we need a table of percentiles of the null distribution of the test statistic. Empirical distribution of the statistic is created for samples generated by a model with $y_0=0$ and $y_t=y_{t-1}+e_t$, $t=1,2,\dots,n$; $n=25$, 50, 100, 250, 500. As in Dickey and Fuller(1981), three replicates of 50,000 samples were generated for $n=25$, two for $n=50,100$, and 250, and one for $n=500$. For each sample size, the 0.01, 0.025, 0.05, 0.10, 0.90, 0.95, 0.975, and 0.99 percentage points of the empirical distribution is provided in Table 1 below.

Table 1. Percentiles of the null distribution of $\hat{\Phi}_s$ for $(\mu, \rho)=(0,1)$ in $y_t=\mu+\rho y_{t-1}+e_t$.

Sample	Probability of a smaller value							
size n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	1.10	1.28	1.48	1.76	6.57	8.21	10.04	12.63
50	1.11	1.30	1.50	1.78	6.29	7.78	9.30	11.46
100	1.09	1.30	1.51	1.79	6.17	7.53	8.94	10.93
250	1.10	1.30	1.52	1.80	6.09	7.45	8.81	10.70
500	1.09	1.31	1.53	1.81	6.09	7.44	8.77	10.65

3. A Monte-Carlo study

In this section, we compare finite sample properties of our symmetric test with the test of Dickey and Fuller(1981) given by $\hat{\Phi}_o=(\bar{\theta}-\theta_0)'(X_o'X_o)(\bar{\theta}-\theta_0)/(2\bar{\sigma}^2)$, where $\bar{\theta}=(X_o'X_o)^{-1}X_o'Y_o$ is the ordinary least squares estimator, $X_o=\{(1,\cdots,1)', (y_1,\cdots,y_{n-1})'\}$, and $Y_o=(y_2,\cdots,y_n)'$. For our study, we consider an AR(1) process $y_t=\mu+\rho y_{t-1}+e_t$, where e_t are independent identically distributed standard normal errors. We consider the parameter configuration: n=25, 50, 100, 250, 500; $\rho=1,0.99,0.95,0.90,0.80$. The observations y_t are simulated using the standard pseudo normal error e_t gennerated by RNNOA in IMSL.

In order to eliminate the start-up effect, y_t, $t=-19,-18,\cdots,n$ are generated and only y_t, $t=1,\cdots,n$ are used in computing the test statistics. We fit the mean-adjusted model (2.4) and compute the symmetric test statistic $\hat{\Phi}_s$.

In Table 2, we report percentages of rejected test statistics for the null hypothesis $(\mu,\rho)=(0,1)$ among 10,000 (or 3,000) number of independent samples. The nominal level is set to 5%.

One interesting result is that the symmetric test is more powerful than the likelihood ratio test for ρ close to 1 or μ close to zero. For the parameter combinations of ρ equal one or μ equal zero, the symmetric test has better power. On the other hand, for μ not close to zero and ρ not close to one, our test lose power severely. A practical implication is discussed in the following section.

Table 2. Empirical powers of $\hat{\Phi}_o$ and $\hat{\Phi}_s$ for nominal level 0.05.

		$\rho=0.8$			$\rho=0.9$			$\rho=0.95$			$\rho=0.99$			$\rho=1.0$		
		μ			μ			μ			μ			μ		
n	Statistic	0.00	0.50	1.00	0.00	0.50	1.00	0.00	0.50	1.00	0.00	0.50	1.00	0.00	0.50	1.00
25	Φ_o	0.07	0.10	0.16	0.05	0.08	0.24	0.04	0.10	0.55	0.04	0.26	0.91	0.05	0.33	0.96
	Φ_s	0.14	0.12	0.08	0.08	0.05	0.06	0.06	0.05	0.51	0.05	0.27	0.95	0.05	0.39	0.97
50	Φ_o	0.24	0.29	0.37	0.08	0.12	0.38	0.05	0.14	0.71	0.04	0.28	0.48	0.05	0.71	1.00
	Φ_s	0.44	0.41	0.29	0.17	0.08	0.02	0.09	0.02	0.38	0.05	0.52	1.00	0.05	0.78	1.00
100	Φ_o	0.79	0.82	0.88	0.23	0.33	0.67	0.08	0.22	0.86	0.05	0.77	1.00	0.05	0.98	1.00
	Φ_s	0.95	0.94	0.92	0.45	0.32	0.12	0.17	0.03	0.03	0.06	0.76	1.00	0.05	0.99	1.00
250	Φ_o	1.00	1.00	1.00	0.93	0.97	1.00	0.35	0.60	0.99	0.05	0.95	1.00	0.05	1.00	1.00
	Φ_s	1.00	1.00	1.00	0.99	0.99	0.97	0.62	0.33	0.05	0.09	0.70	1.00	0.05	1.00	1.00
500	Φ_o	1.00	1.00	1.00	1.00	1.00	1.00	0.92	0.98	1.00	0.08	0.98	1.00	0.05	1.00	1.00
	Φ_s	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.98	0.87	0.18	0.12	1.00	0.05	1.00	1.00

For $\mu=0$ power is computed from 10,000 samples.
For $\mu\neq0$ power is computed from 3,000 samples.

4. Conclusion

In the Monte-Carlo simulation, our symmetric test is shown to be more powerful than the likelihood ratio test in the vicinity of the null hypothesis. This is an encouraging result because difficulty in hypothesis testing lies for the parameter close to the null. We can use the symmetric test as a supplementary tool to the likelihood ratio test. When the likelihood ratio test does not reject the null hypothesis, one may see also the symmetric test. If the symmetric test also does not reject the random walk hypothesis, we can say more confidently that the series is a random walk.

When the likelihood ratio test does not reject the random walk hypothesis, owning to the relative lower power of the likelihood ratio test in the vicinity of the null, there is still possibility that the series under study is not a random walk. In that case, our test can be used as a cross check. If our test also does not reject the null hypothesis, we get more confidence for the random walk hypothesis than what the likelihood ratio test gives us.

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