The Korean Communications
in Statistics Vol. 4, No. 2, 1997
pp. 541 - 548

# Exact Confidence Intervals on the Regression Coefficients in Multiple Regression Model with Nested Error Structure

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## Abstract

In regression model with nested error structure interval estimations on regression coefficients in different stages are proposed. Ordinary least square estimators and generalized least square estimators of the regression coefficients in this model are derived for between and within group model. The confidence intervals are derived by using independent distributional properties between regresssion coefficient estimators and quadratic forms obtained from the model.

## 1. Introduction

This article considers the multiple regression model where the responses are correlated. This model is appropriate to the data collected using two stage cluster designs, panel data analysis, or split-plot type models. Fuller and Battese(1973) presented transformation by which uncorrelated errors with constant variances were obtained and the transformations permited the calculation of the Generalized Least Squares(GLS) estimators and their covariance matrices by Ordinary Least Squares(OLS) regression. Christensen(1987) analyzed two stage sampling data and showed OLS estimators were Best Linear Unbiased Estimators(BLUE) iff for each variable in the model, variable obtained by replacing each component with corresponding cluster average was also contained in the model. Weerakkoday and Johnson(1992) presented sufficient conditions under which OLS estimates of regressor parameters in nested error data structure were Uniformly Minimum Variance Unbiased(UMVU). Park and Burdick(1994) proposed the confidence intervals on the regression coefficients in simple regression model with one-fold nested error structure. Park(1996) proposed the distributional properties of variance components in two stage regression model. This article proposes the exact confidence intervals on the regression coefficients associated with primary and secondary units in multiple regression model with nested error structure.

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## 2. Multiple regression model with nested error structure

The multiple regression model with nested error structure is written as

$$\begin{aligned} Y_{ij} &= \beta_0 + \beta_1 X_{h1} + \cdots + \beta_{p_1} X_{hp_1} + \delta_i \\ &\quad + \gamma_1 X_{ij1} + \cdots + \gamma_{p_2} X_{ijp_2} + \varepsilon_{ij} \\ h &= 1, \cdots, m_{p_1};\ i = 1, \cdots, l_1;\ j = 1, \cdots, l_2 \end{aligned} \tag{2.1}$$

where $Y_{ij}$ is the $j$ th observation in the $i$ th cell(group), $\beta_0$ is an intercept term, $\beta_1, \cdots, \beta_{p_1}$ are unknown parameters associated with primary units, $X_{h1}, \cdots, X_{hp_1}$ are fixed predictor variables in the primary unit, $\gamma_1, \cdots, \gamma_{p_2}$ are unknown parameters associated with secondary units, $X_{ij1}, \cdots, X_{ijp_2}$ are fixed predictor variables in the secondary unit, $\delta_i$ is a random error term in the primary unit, $\varepsilon_{ij}$ is a random error term in the secondary unit, $\delta_i$ and $\varepsilon_{ij}$ are jointly independent normal random variables with zero means and variances $\sigma_\delta^2$ and $\sigma_\varepsilon^2$, respectively. The index $l_1$ is the number of different combinations(cells) of levels among $X_{ij}$'s, i.e., $l_1 = m_1 \times m_2 \times \cdots \times m_{p_1}$ and $l_2$ is the number of repetitions within an $i$ th cell. We consider the balanced case where $l_2$'s are same for all $i$'s. Since $\beta$'s, $\gamma$'s, $X_{ij}$'s, and $X_{ijk}$'s are fixed, and $\delta_i$ and $\varepsilon_{ij}$ are random, model (2.1) is a mixed model.

The model (2.1) is written in matrix notation,

$$\underline{Y} = ZX_1\underline{\beta} + X_2\underline{\gamma} + Z\underline{\delta} + \underline{\varepsilon} \tag{2.2}$$

where $\underline{Y}$ is an $l_1 l_2 \times 1$ vector of observations, $Z$ is an $l_1 l_2 \times l_1$ design matrix with 0's and 1's, i.e., $Z = \bigoplus_{i=1}^{l_1} 1_{l_2}$ where $1_{l_2}$ is an $l_2 \times 1$ column vector of 1's and $\oplus$ is the direct sum operator, $X_1$ is an $l_1 \times (1+p_1)$ matrix of known values with a column of 1's in the first column and $p_1$ columns of $X_{ij}$'s from the second column to the $p_1$ th column, $\underline{\beta}$ is a $(1+p_1) \times 1$ vector of parameters associated with $X_{ij}$'s, $X_2$ is an $l_1 l_2 \times p_2$ matrix of known values with $p_2$ columns of $X_{ijk}$'s from the first column to the $p_2$ th column, $\underline{\gamma}$ is a $p_2 \times 1$ vector of parameters associated with $X_{ijk}$'s, $\underline{\delta}$ is an $l_1 \times 1$ vector of random error terms, and $\underline{\varepsilon}$ is an $l_1 l_2 \times 1$ vector of random error terms. Refer to Park(1996) for specific matrix forms in detail. By the assumptions in (2.1) the response variables are normally distributed

$$\underline{Y} \sim N(X\underline{\alpha},\ \sigma_\delta^2 ZZ' + \sigma_\varepsilon^2 I_{l_1 l_2}) \tag{2.3}$$

where

$$X = (\ ZX_1\ \ X_2\ ) \text{ and } \underline{\alpha} = \begin{pmatrix} \underline{\beta} \\ \underline{\gamma} \end{pmatrix}.$$

Since $Z'Z/l_2 = I_{l_1}$, premultiplying (2.2) by $Z'/l_2$ gives

$$\underline{\overline{Y}} = X_1\underline{\beta} + X_2^*\underline{\gamma} + \underline{\delta} + \underline{\varepsilon}^* \tag{2.4}$$

where $\underline{\overline{Y}} = (Z'/l_2)\underline{Y}$, $X_2^* = (Z'/l_2)X_2$, and $\underline{\varepsilon}^* = (Z'/l_2)\underline{\varepsilon}$.

The reductions in sums of squares of the model are attributable to fitting the primary and secondary fixed variables and expressed into the quadratic forms. Let $G_1 = (X^{*'}X^*)^{-1}$ and $G_2 = (\overline{X_2}'\ \overline{X_2})^{-1}$ where $X^* = (X_1\ X_2^*)$, $X_2^* = (Z'/l_2)X_2$, $\overline{X_2} = WX_2$, and $W = I_{l_1l_2} - ZZ'/l_2$. Define $H_1 = X^* G_1 X^{*'}$ and $H_2 = \overline{X_2}\, G_2 \overline{X_2}'$. Now consider the quadratic forms $R_1 = \underline{Y}'(Z/l_2)(I_{l_1} - H_1)(Z'/l_2)\underline{Y}$ and $R_2 = \underline{Y}'W'(I_{l_1l_2} - H_2)W\underline{Y}$. Under the distributional assumptions in (2.1) the quadratic forms $R_1/(\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2))$ and $R_2/\sigma_\varepsilon^2$ are chi-squared random variables with $l_1 - p_1 - p_2 - 1$ and $l_1l_2 - l_1 - p_2$ degrees of freedom, repectively. In addition, two quadratic forms are independent(see Park(1996)).

## 3. Exact confidence intervals for regression coefficients

In order to construct confidence intervals on the regression coefficients the distributional properties of the estimators of $\underline{\beta}$ and $\underline{\gamma}$ are examined and they are summarized in theorems in this section. The exact confidence intervals for regression coefficients in the model are proposed using the theorems.

### 3.1 Confidence interval on $\gamma_i$ using within group OLS estimator

Since $WZ = 0$, multiplying both sides of (2.2) on the left by $W$ gives

$$\underline{\tilde{Y}} = \overline{X_2}\underline{\gamma} + W\underline{\varepsilon} \tag{3.1}$$

where $\underline{\tilde{Y}} = W\underline{Y}$. Notice that $W$ is a symmetric and idempotent matrix. Since $\underline{\tilde{Y}}$ is a linear combination of $\underline{Y}$, $\underline{\tilde{Y}}$ is normally distributed

$$\underline{\tilde{Y}} \sim N(\overline{X_2}\underline{\gamma},\ \sigma_\varepsilon^2 W). \tag{3.2}$$

The OLS estimators for $\underline{\gamma}$ from the within model are expessed as $p_2$ elements of the vector

$$\underline{\hat{\gamma}}_W = (\overline{X_2}'\overline{X_2})^{-1}\overline{X_2}'\underline{\tilde{Y}}. \tag{3.3}$$

The estimators $\underline{\hat{\gamma}}_W$ are called the within group OLS estimators of $\underline{\gamma}$ because only the variation within each group is utilized . Since $\underline{\hat{\gamma}}_W$ are a linear combination of $\underline{Y}$,

$$\underline{\hat{\gamma}}_W \sim N(\underline{\gamma},\ \sigma_\varepsilon^2(\overline{X_2}'\overline{X_2})^{-1}). \tag{3.4}$$

**Theorem 3.1** Under the distributional assumptions in (2.1), $\underline{\hat{\gamma}}_W$ and $R_2$ are independent.

**Proof** $R_2$ is written as $R_2 = \underline{Y}'W'(I_{l_1l_2} - H_2)W\underline{Y} = \underline{\tilde{Y}}'(I_{l_1l_2} - H_2)\underline{\tilde{Y}}$. Note that $(\overline{X}_2'\overline{X}_2)^{-1}\overline{X}_2'(\sigma_\epsilon^2 W)(I_{l_1l_2} - H_2) = 0$ since $\overline{X}_2'W = \overline{X}_2'$, $WH_2 = H_2$ and $\overline{X}_2'H_2 = \overline{X}_2'$. It follows by Theorem 3(Searle 1971, p59) that $\underline{\hat{\gamma}}_W$ and $R_2$ are independent.

Note that $(I_{l_1l_2} - H_2)$ in $R_2$ is an idempotent matrix. Therefore, the within group OLS estimators of regression coefficients associated with secondary units in the model are $t$ distributed from (3.4) and Theorem 3.1. That is, it follows from (3.4) that

$$\frac{\hat{\gamma}_{iW} - \gamma_i}{\sqrt{\overline{X}_2^{ii}\sigma_\epsilon^2}} \sim N(0,1)$$

where $\hat{\gamma}_{iW}$ is the $i$ th OLS estimator of regression coefficients in the secondary unit based on within group model and $\overline{X}_2^{ii}$ is the $i$ th diagonal element of matrix $(\overline{X}_2'\overline{X}_2)^{-1}$. It follows from Theorem 3.1 that

$$\frac{\hat{\gamma}_{iW} - \gamma_i}{\sqrt{\overline{X}_2^{ii}\sigma_\epsilon^2}} \div \sqrt{\frac{R_2}{\sigma_\epsilon^2(l_1l_2 - l_1 - p_2)}} \sim t(l_1l_2 - l_1 - p_2). \tag{3.5}$$

Therefore, the proposed exact $1-2\alpha$ two-sided confidence interval on $\gamma_i$ is

$$\hat{\gamma}_{iW} \pm t_{\alpha:l_1l_2-l_1-p_2}\sqrt{\overline{X}_2^{ii}S_\epsilon^2} \tag{3.6}$$

where $S_\epsilon^2 = R_2/(l_1l_2 - l_1 - p_2)$ and $t_{\delta:v}$ is the $1-\delta$ percentile $t$-value with $v$ degrees of freedom. This method is referred to as EXCW method.

### 3.2 Confidence interval on $\gamma_i$ using between group OLS estimator

From the fact that $Z'Z/l_2 = I_{l_1}$ and (2.4), $\overline{\underline{Y}}$ are normally distributed

$$\overline{\underline{Y}} \sim N(X_1\underline{\beta} + X_2^*\underline{\gamma},\ (\sigma_\delta^2 + (\sigma_\epsilon^2/l_2))I_{l_1}) \tag{3.7}$$

The OLS estimators for $\underline{\beta}$ and $\underline{\gamma}$ from the between model are denoted by $\underline{\hat{\beta}}_A$ and $\underline{\hat{\gamma}}_A$, respectively, and they are obtained from (2.4). The estimators $\underline{\hat{\beta}}_A$ and $\underline{\hat{\gamma}}_A$ are called the between group OLS estimators for the first and second stage predictor variables, respectively, since they ignore variation within group. Normal equations for (2.4) are

$$\begin{pmatrix} X_1' \\ X_2^{*\prime} \end{pmatrix}(X_1\ X_2^*)\begin{pmatrix} \underline{\hat{\beta}}_A \\ \underline{\hat{\gamma}}_A \end{pmatrix} = \begin{pmatrix} X_1' \\ X_2^{*\prime} \end{pmatrix}\overline{\underline{Y}}. \tag{3.8}$$

From normal equations (3.8) and Formula (78) (Searle 1987, p263) $\underline{\hat{\gamma}}_A$ are expressed as $p_2$ elements of the vector

$$\underline{\hat{\gamma}}_A = (X_2^{*\prime}M_1X_2^*)^{-1}X_2^{*\prime}M_1\overline{\underline{Y}} \tag{3.9}$$

where $M_1 = (I_{l_1} - X_1(X_1'X_1)^{-1}X_1')$. By elementary algebra $\hat{\gamma}_A$ are normally distributed

$$\hat{\gamma}_A \sim N(\gamma, \ (\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2))[X_2^{*'} M_1 X_2^*]^{-1}). \tag{3.10}$$

**Theorem 3.2** Under the distributional assumptions in (2.1), $\hat{\gamma}_A$ and $R_1$ are independent.

**Proof** $R_1$ is written as $R_1 = \underline{Y}'(Z/l_2)(I_{l_1} - H_1)(Z'/l_2)\underline{Y} = \overline{\underline{Y}}'(I_{l_1} - H_1)\overline{\underline{Y}}$. Note that $X_2^{*'} M_1(\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2)) I_{l_1}(I_{l_1} - H_1) = 0$ since $X_1'H_1 = X_1'$ and $X_2^{*'}H_1 = X_2^{*'}$. Therefore, $\hat{\gamma}_A$ and $R_1$ are independent.

Note that $(I_{l_1} - H_1)$ in $R_1$ is an idempotent matrix. Similar reasoning in (3.5) is applied to obtain an exact confidence interval. From (3.10) and Theorem 3.2 the between group OLS estimators of regression coefficients associated with secondary units in model are $t$ distributed. Therefore, the proposed exact $1-2\alpha$ two-sided confidence interval on $\gamma_i$ is

$$\hat{\gamma}_{iA} \pm t_{\alpha:l_1-p_1-p_2-1}\sqrt{\overline{X_2^{*}}^{ii} S_\delta^2} \tag{3.11}$$

where $\hat{\gamma}_{iA}$ is the $i$ th OLS estimator of regression coefficients in the secondary unit based on between group model and $\overline{X_2^*}^{ii}$ is the $i$ th diagonal element of matrix $(X_2^{*'} M_1 X_2^*)^{-1}$. This method is referred to as EXCA method.

### 3.3 Confidence interval on $\beta_i$ using between group OLS estimator

From normal equations (3.8) and formula (77) (Searle 1987, p263) $\hat{\beta}_A$ are expressed as $(1+p_1)$ elements of the vector

$$\hat{\beta}_A = (X_1'X_1)^{-1}X_1'(\overline{\underline{Y}} - X_2^* \hat{\gamma}_A). \tag{3.12}$$

From (3.7) and (3.10) $Var(\hat{\beta}_A)$ is written as

$$\begin{aligned}
Var(\hat{\beta}_A) &= (X_1'X_1)^{-1}X_1'\{Var(\overline{\underline{Y}}) + Var(X_2^* \hat{\gamma}_A) - 2COV(\overline{\underline{Y}}, X_2^* \hat{\gamma}_A)\}X_1(X_1'X_1)^{-1} \\
&= (X_1'X_1)^{-1}X_1'\{Var(\overline{\underline{Y}}) + Var(X_2^* \hat{\gamma}_A)\}X_1(X_1'X_1)^{-1} \\
&\quad -2(X_1'X_1)^{-1}X_1'COV(\overline{\underline{Y}}, X_2^*(X_2^{*'}M_1X_2^*)^{-1}X_2^{*'}M_1\overline{\underline{Y}})X_1(X_1'X_1)^{-1} \\
&= (X_1'X_1)^{-1}X_1'\{(\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2)) \\
&\quad +(\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2))X_2^*(X_2^{*'}M_1X_2^*)^{-1}X_2^{*'}\}X_1(X_1'X_1)^{-1} \\
&\quad -2(\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2))(X_1'X_1)^{-1}X_1'M_1'X_2^*(X_2^{*'}M_1X_2^*)^{-1}X_2^{*'}X_1(X_1'X_1)^{-1} \\
&= (\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2))\{(X_1'X_1)^{-1} \\
&\quad +(X_1'X_1)^{-1}X_1'X_2^*[X_2^{*'}M_1X_2^*]^{-1}X_2^{*'}X_1(X_1'X_1)^{-1}\}
\end{aligned}$$

by using that $M_1$ is symmetric and $X_1'M_1 = 0$. Since $\hat{\beta}_A$ are also a linear combination of $\underline{Y}$, $\hat{\beta}_A$ are normally distributed

$$\hat{\beta}_A \sim N(\beta,\ V(\hat{\beta}_A)) \tag{3.13}$$

where

$$V(\hat{\beta}_A) = (\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2))\{(X_1'X_1)^{-1} + (X_1'X_1)^{-1}X_1'X_2^*[X_2^{*'}M_1X_2^*]^{-1}X_2^{*'}X_1(X_1'X_1)^{-1}\}.$$

**Theorem 3.3** Under the distributional assumptions in (2.1), $\hat{\beta}_A$ and $R_1$ are independent.

**Proof** $\hat{\beta}_A$ are written as $\hat{\beta}_A = (X_1'X_1)^{-1}X_1'(\overline{Y} - X_2^*\hat{\gamma}_A) = (X_1'X_1)^{-1}X_1'(I_{l_1} - X_2^*[X_2^{*'}M_1X_2^*]^{-1}X_2^{*'}M_1)\overline{Y}$ by using (3.9). Note that $(X_1'X_1)^{-1}X_1'(I_{l_1} - X_2^*[X_2^{*'}M_1X_2^*]^{-1}X_2^{*'}M_1)(\sigma_\delta^2 + (\sigma_\varepsilon^2/l_2))I_{l_1}(I_{l_1} - H_1) = 0$ since $X_1'H_1 = X_1'$, $(X_1'X_1)^{-1}X_1'(I_{l_1} - H_1) = 0$, $M_1(I_{l_1} - H_1) = I_{l_1} - H_1$, and $X_2^{*'}H_1 = X_2^{*'}$. Thus $\hat{\beta}_A$ and $R_1$ are independent.

Theorem 3.3 and (3.13) are used to construct a confidence interval on the first stage regression coefficients in the model. Similar reasoning in (3.5) is applied to obtain an exact confidence interval. The between group OLS estimators of regression coefficients associated with primary units in the model are $t$ distributed. Thus, the proposed exact $1-2\alpha$ two-sided confidence interval on $\beta_i$ is

$$\hat{\beta}_{iA} \pm t_{\alpha:l_1-p_1-p_2-1}\sqrt{X^{*ii}S_\delta^2} \tag{3.14}$$

where $\hat{\beta}_{iA}$ is the $i$ th OLS estimator of regression coefficients in the primary units based on between group model, $X^{*ii}$ is the $i$ th diagonal element of matrix $\{(X_1'X_1)^{-1} + (X_1'X_1)^{-1}X_1'X_2^*[X_2^{*'}M_1X_2^*]^{-1}X_2^{*'}X_1(X_1'X_1)^{-1}\}$, and $S_\delta^2 = R_1/(l_1 - p_1 - p_2 - 1)$. This method is referred to as EXB method.

### 3.4 Confidence interval on $\gamma_i$ using within group GLS estimator

The GLS estimators for $\gamma$ from the within model, $\hat{\gamma}_{WG}$, are expressed as $p_2$ elements of the vector and are written as

$$\hat{\gamma}_{WG} = (\overline{X_2}'W^{-1}\overline{X_2})^{-1}\overline{X_2}'W^{-1}\underline{\tilde{Y}} \tag{3.15}$$

since $V_{\underline{\tilde{Y}}} = Var(\underline{\tilde{Y}}) = \sigma_\varepsilon^2 W$. In addition, the GLS estimators, $\hat{\gamma}_{WG}$, are normally distributed

$$\hat{\gamma}_{WG} \sim N(\gamma,\ \sigma_\varepsilon^2(\overline{X_2}'W^{-1}\overline{X_2})^{-1}). \tag{3.16}$$

**Theorem 3.4** Under the distributional assumptions in (2.1), $\hat{\gamma}_{WG}$ and $R_2$ are independent.

**Proof** Note that $R_2 = \underline{Y}'(I_{l_1 l_2} - H_2)\underline{Y}$. Thus, $(\overline{X_2}' W^{-1} \overline{X_2})^{-1} \overline{X_2}' W^{-1} (\sigma_\epsilon^2 W)(I_{l_1 l_2} - H_2) = 0$ since $\overline{X_2}' H_2 = \overline{X_2}'$ It follows that $\hat{\underline{\gamma}}_{WG}$ and $R_2$ are independent.

Similar reasoning in (3.5) is applied to obtain an exact confidence interval. The GLS estimators of regression coefficients associated with secondary units in the model are $t$ distributed from (3.16) and Theorem 3.4. Therefore, the proposed exact $1-2\alpha$ two-sided confidence interval on $\gamma_i$ is

$$\hat{\gamma}_{iWG} \pm t_{\alpha: l_1 l_2 - l_1 - p_2} \sqrt{\overline{X_w^*}^{ii} S_\epsilon^2} \tag{3.17}$$

where $\hat{\gamma}_{iWG}$ is the $i$ th GLS estimator of regression coefficients in the secondary unit based on within model and $\overline{X_w^*}^{ii}$ is the $i$ th diagonal element of matrix $(\overline{X_2}' W^{-1} \overline{X_2})^{-1}$. This method is referred to as EXCG method.

The GLS estimators for $\underline{\beta}$ and $\underline{\gamma}$ from the between model are same as $\hat{\underline{\beta}}_A$ and $\hat{\underline{\gamma}}_A$, respectively, since $Var(\overline{\underline{Y}}) = (\sigma_\delta^2 + (\sigma_\epsilon^2 / l_2)) I_{l_1}$ in (3.7). In particular, The GLS estimators for $\underline{\gamma}$ from the between model, $\hat{\underline{\gamma}}_{AG}$, are written

$$\hat{\underline{\gamma}}_{AG} = (X_2^{*'} M_1 X_2^*)^{-1} X_2^{*'} M_1 \overline{\underline{Y}} \tag{3.18}$$

which are same as $\hat{\underline{\gamma}}_A$ in (3.9). Similarly, the GLS estimators for $\underline{\beta}$ from the between model, $\hat{\underline{\beta}}_{AG}$, are equivalent to $\hat{\underline{\beta}}_A$ in (3.12).

## 4. Conclusions

This paper used OLS and GLS estimators to find regression coefficients in the model. The estimators, $\hat{\underline{\beta}}$ and $\hat{\underline{\gamma}}$, turned out to be independent of quadratic forms, $R_1$ and $R_2$, obtained from the model. One exact two-sided confidence interval on $\beta_i$ and three exact two-sided confidence interval on $\gamma_i$ were proposed. EXB method is recommended to construct confidence interval on regression coefficients associated with the primary unit in the model. EXCW, EXCA, and EXCG methods are recommended to construct confidence intervals on regression coefficients associated with the secondary unit in the model. The proposed confidence intervals on regression coefficients could be used in two stage regression model application areas.

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