The Korean Communications
in Statistics Vol. 4, No. 2, 1997
pp. 523 - 532

Bootstrap Confidence Intervals for the Reliability Function of an Exponential Distribution

Suk-Bok Kang[1] and Young-Suk Cho[2]

Abstract

We propose several estimators of the reliability function R of the two-parameter exponential distribution, and then compare those estimators in terms of the mean square error (MSE) through Monte Carlo method. We also consider the parametric bootstrap estimation. Using the parametric bootstrap estimator, we obtain the bootstrap confidence intervals for the reliability function and compare the proposed bootstrap confidence intervals in terms of the length and the coverage probability through Monte Carlo method.

1. Introduction

The mathematical theory of the reliability has grown out of the demands of modern technology and particularly out of the experiences in World War II with complex military systems. One of the first area of the reliability to be approached with any mathematical sophistication was the area of machine maintenance. The earliest attemps to justify the Poisson distribution as the input distribution of calls to a telephone trunk also laid the basis for using the exponential as the failure law of complex equipment.

In life testing research, the simplest and the most widely exploited model is the two-parameter exponential distribution with p.d.f.

$$f(x;\theta,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{(x-\theta)}{\sigma}\right),\ 0<\ \theta<\ x,\ 0\ <\ \sigma, \tag{1.1}$$

where θ and σ are the location and the scale parameters, respectively.

In the study of life testing and reliability analysis one important approach has been to consider an underlying life distribution and to find suitable estimates of the parameters of that distribution. Epstein and Sobel (1954) published a paper that presented the maximum likelihood estimators (MLEs) of the scale and the location parameters in the two-parameter exponential

1) Associate Professor, Department of Statistics, Yeungnam University, Kyongsan, 712-749, Korea
2) Lecturer, Department of Statistics, Yeungnam University, Kyongsan, 712-749, Korea

distribution. Singh and Gupta (1993) studied the estimation of the parameters of an exponential distribution when the parameter space is restricted. Sinha (1985) obtained the Bayes estimates of the reliability function in a normal distribution. Woo and Kang (1986) considered the jackknife estimation of the reliability function in a Poisson distribution. Kang (1987) considered the jackknife estimation of the reliability function in a exponential distribution when the scale parameter is known. Abedin and Karson (1993) studied shrunken estimators and Bayes estimators of the reliability function in the one parameter exponential distribution when observations are time censored and replaced.

The bootstrap method, introduced by Efron (1979), is resampling technique and a very general method to create measures of uncertainty and bias, in particular at parameter estimation from independent identically distributed variables. Efron (1981, 1982, 1985, 1987) has introduced and refined the percentile method of using bootstrap caculations to set approximate confidence limits for parameters. These refinements of the percentile method are the bias corrected (BC) percentile method and the accelerated bias-corrected (BC_a) percentile method. The bootstrap method and other methods for assessing statistical accuracy are summarized by Efron and Tishirani (1993). Recently, the theoretical properties of the jackknife and bootstrap methods are summarized by Shao and Tu (1995). Kang and Cho (1997) proposed the nonparametric bootstrap confidence intervals after bias correcting.

In section 2, we compare the estimators of R in terms of the MSE through Monte Carlo method and obtain the parametric bootstrap estimator of R. Using the proposed bootstrap estimator, we obtain the bootstrap confidence intervals for R and compare the bootstrap confidence intervals through Monte Carlo method, and summarize the numerical results.

2. Parametric Bootstrap Confidence Intervals

Let $X_1, X_2, \ldots, X_n$ be a random sample from two-parameter exponential distribution (1.1). These random variables represent the life-lengths of n identical systems and $X_{(1)}$, $X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics. The reliability function for given time t in two-parameter exponential distribution is given by

$$R(t;\theta,\sigma)=\begin{cases}\exp\left(\frac{-(t-\theta)}{\sigma}\right), & t \geq \theta \\ 1, & t < \theta.\end{cases} \tag{2.1}$$

Since the MLEs of σ and θ are given by

$$\hat{\sigma}_{MLE} = \sum_{i=1}^{n}(X_i - X_{(1)})/n$$
$$\hat{\theta}_{MLE} = X_{(1)}.$$

The MLE $\hat{R}_{MLE}(t;\theta,\sigma)$ of the reliability function is given by

$$\hat{R}_{MLE} = \begin{cases} \exp\left(-\dfrac{n(t-X_{(1)})}{\sum_{i=1}^{n}(X_i-X_{(1)})}\right), & t \geq X_{(1)} \\ 1, & t < X_{(1)}. \end{cases} \tag{2.2}$$

Now, we propose several estimators of R using the several estimators of the two parameters θ and σ.

First, by using the UMVUE $\hat{\sigma}_U = \sum_{i=1}^{n}(X_i - X_{(1)})/(n-1)$ of σ and the minimum risk estimator (MRE) $\hat{\theta}_M = (n+1)X_{(1)}/n - \sum_{i=1}^{n} X_i/n^2$ of θ among the class of the estimators of the form $c_1 X_{(1)} + c_2 \sum_{i=1}^{n}(X_i - X_{(1)})/n$ (c_1 and c_2 are constants), the estimator $\hat{R}_{MU}(t;\theta,\sigma)$ is given by

$$\hat{R}_{MU}(t;\theta,\sigma) = \begin{cases} \exp\left(-\dfrac{(n-1)(n^2 t + \sum_{i=1}^{n} X_i - n(n+1)X_{(1)})}{n^2 \sum_{i=1}^{n}(X_i - X_{(1)})}\right), & t \geq \hat{\theta}_M \\ 1, & t < \hat{\theta}_M. \end{cases} \tag{2.3}$$

Second, by using the MLE $\hat{\sigma}_{MLE}$ of σ and the MRE $\hat{\theta}_M$ of θ, the estimator of $\hat{R}_{MM}(t;\theta,\sigma)$ is given by

$$\hat{R}_{MM}(t;\theta,\sigma) = \begin{cases} \exp\left(-\dfrac{n^2 t + \sum_{i=1}^{n} X_i - n(n+1)X_{(1)}}{n \sum_{i=1}^{n}(X_i - X_{(1)})}\right), & t \geq \hat{\theta}_M \\ 1, & t < \hat{\theta}_M. \end{cases} \tag{2.4}$$

Third, by using the UMVUE $\hat{\sigma}_U$ of σ and the UMVUE $\hat{\theta}_U = nX_{(1)}/(n-1) - \sum_{i=1}^{n} X_i/n(n-1)$ of θ, the estimator $\hat{R}_{UU}(t;\theta,\sigma)$ is given by

$$\hat{R}_{UU}(t;\theta,\sigma) = \begin{cases} \exp\left(-\dfrac{n(n-1)t + \sum_{i=1}^{n} X_i - n^2 X_{(1)}}{n \sum_{i=1}^{n}(X_i - X_{(1)})}\right), & t \geq \hat{\theta}_U \\ 1, & t < \hat{\theta}_U. \end{cases} \tag{2.5}$$

Theorem 2.1. The proposed estimators of R are consistent estimators.

Proof. Since $E[(\hat{\theta}_{MLE}-\theta)^2]=2\sigma^2/n^2$, $E[(\hat{\theta}_U-\theta)^2]=\sigma^2/(n-1)n$, $E[(\hat{\sigma}_{MLE}-\sigma)^2]=\sigma^2/n$, $E[(\hat{\sigma}_U-\sigma)^2]=\sigma^2/(n-1)$, and $E[(\hat{\theta}_M-\theta)^2]=(n+1)\sigma^2/n^3$, the parametric estimators of the location and the scale parameters in an exponential distribution are convergence in propability to parameter. From these results, Slutsky's theorem, and the fact that e^{-x} is continuous function, this completes the proof.

From (2.2), (2.3), (2.4), (2.5), we calculate the mean squared errors of the estimators for sample size n = 10(20)50(based on 2,000 Monte Carlo runs) when the location parameter $\theta=1$ and the scale parameter $\sigma=0.5(0.5)1.5$, and $t=1.2,\ 2$. These values are given in Table 1. From Table 1, the estimator $\hat{R}_{MU}$ is more efficient than the other estimators in the sense of the MSE. So we propose the bootstrap estimator $\hat{R}_{MU}^*$ of R as follows;

$$\hat{R}_{MU}^*(b)=\begin{cases}\exp\left(-\dfrac{t-\hat{\theta}_M^*(b)}{\hat{\sigma}_U^*(b)}\right), & t\geq\ \hat{\theta}_M^*(b)\\ 1, & t<\ \hat{\theta}_M^*(b).\end{cases}$$

Theorem 2.2. The bootstrap estimator $\hat{R}_{MU}^*$ is a consistent estimator of R.

Proof. For arbitrary positive ε,

$$\begin{aligned}P[|\ \hat{\sigma}_U^*-\ \hat{\sigma}_{MLE}|\geq\varepsilon] &\leq\frac{E(\ \hat{\sigma}_U^*-\ \hat{\sigma}_{MLE})^2}{\varepsilon^2}\\ &=\frac{E[E_*[(\ \hat{\sigma}_U^*-\ \hat{\sigma}_{MLE})^2|X_1,X_2,\ldots,X_n\ ]]}{\varepsilon^2}\\ &=\frac{E(\ \hat{\sigma}_{MLE}^2\ )}{(n-1)\varepsilon^2}\\ &=\frac{\sigma^2}{n\varepsilon^2}\rightarrow 0,\ as\ n\rightarrow\infty\end{aligned}$$

and

$$\begin{aligned}P[|\ \hat{\theta}_M^*-\ \hat{\theta}_M|\geq\varepsilon\ ] &\leq\frac{E(\ \hat{\theta}_M^*-\ \hat{\theta}_M\ )^2}{\varepsilon^2}\\ &=\frac{E[\ E_*[(\ \hat{\theta}_M^*-\ \hat{\theta}_M\ )^2|X_1,X_2,\ldots,X_n\ ]]}{\varepsilon^2}\\ &=\frac{(n+1)E(\ \hat{\sigma}_{MLE}^2\ )}{n^3\varepsilon^2}\\ &=\frac{(n^2-1)\sigma^2}{n^4\varepsilon^2}\rightarrow 0,\ as\ n\rightarrow\infty,\end{aligned}$$

where $E_*[\ \cdot\ |X_1,X_2,\ldots,X_n]$ denotes the conditional expectation for given

$X_1, X_2, \ldots, X_n$. Therefore, $\hat{\sigma}_U^*$ and $\hat{\theta}_M^*$ converge in propability to σ and θ, respectively. Since e^{-x} is continuous function, $\hat{R}_{MU}^*$ converge in propability to R.

We can not obtian the exact confidence interval for R. So we can obtian the approximate confidence intervals for R by using the bootstrap confidence intervals and the parametric bootstrap estimator.

The parametric bootstrap method algorithm is the following. Among the several estimators of the scale and the location parameters, $\hat{\theta}_M$ and $\hat{\sigma}_{MLE}$ are more efficent than the other estimators in the sense of the MSE. So we select B independent bootstrap samples $X^{*1}, X^{*2}, \ldots, X^{*B}$ each consisting of n data values which are generated from $F(\hat{\theta}_M, \hat{\sigma}_{MLE})$ and evalua te the bootstrap replication corresponding to each bootstrap sample $(b=1, 2, \ldots, B)$, $\hat{\theta}_M^*(b) = \hat{\theta}_M(X^{*b})$ and $\hat{\sigma}_U^*(b) = \hat{\sigma}_U(X^{*b})$.

By the parametric bootstrap estimator $\hat{R}_{MU}^*$ of R, we propose the parametric bootstrap confidence intervals for R. Let $\hat{G}$ be the cumulative distribution function of $\hat{R}_{MU}^*$. Then the $100(1-2\alpha)\%$ percentile interval is defined by the α and $1-\alpha$ percentiles of $\hat{G}$;

$$[\hat{R}_{\%,lo}, \ \hat{R}_{\%,up}] = [\hat{G}^{-1}(\alpha), \ \hat{G}^{-1}(1-\alpha)].$$

Since by definition $\hat{G}^{-1}(\alpha) = \hat{R}_{MU}^*(\alpha)$ the $100 \cdot \alpha$th percentile of the bootstrap distribution, the percentile interval (PI) can be written as

$$[\hat{R}_{\%,lo}, \ \hat{R}_{\%,up}] = [\hat{R}^*_{MU}(\alpha), \ \hat{R}^*_{MU}(1-\alpha)]. \tag{2.6}$$

From the assumption that

$$T = \frac{\hat{R}_{MU} - R}{\widehat{se}(\hat{R})} \approx t_{n-1},$$

where t_{n-1} represents the Student's t distribution with $n-1$ degrees of freedom and $\widehat{se}(\hat{R}_{MU})$ is the bootstrap estimator of standard error of $\hat{R}_{MU}$, the t interval (STI) is given by

$$[\hat{R}_{\%,lo}, \ \hat{R}_{\%,up}] = [\hat{R}_{MU} - t_{n-1}(1-\alpha) \cdot \widehat{se}, \ \hat{R}_{MU} - t_{n-1}(\alpha) \cdot \widehat{se}], \tag{2.7}$$

where $t_{n-1}(\alpha)$ is the $100 \cdot \alpha$th percentile of the t distribution with $n-1$ degrees of freedom.

The bias-corrected and accelerated (BC_a) interval is a substantial improvement over the percentile interval in both theory and practice. The $100(1-2\alpha)\%$ BC_a interval is given by

$$[\ \hat{R}_{\%,lo},\ \hat{R}_{\%,up}\ ]=[\ \hat{R}^*_{MU}(\alpha_1),\ \hat{R}^*_{MU}(\alpha_2)], \tag{2.8}$$

where

$$\alpha_1 = \Phi\left(\hat{z}_0+\frac{\hat{z}_0+z(\alpha)}{1-\hat{a}(\hat{z}_0+z(\alpha))}\right)$$
$$\alpha_2 = \Phi\left(\hat{z}_0+\frac{z_0+z(1-\alpha)}{1-\hat{a}(z_0+z(1-\alpha))}\right)$$

and $\Phi(\cdot)$ is the standard normal cumulative distribution function, and $z(\alpha)$ is the $100\cdot\alpha$th percentile point of a standard normal distribution. The value of the bias-correction $\hat{z}_0$ is obtained directly from the proportion of the bootstrap replications less than the original estimator $\hat{R}_{MU}$,

$$\hat{z}_0=\Phi^{-1}\left(\frac{\#\{\ \hat{R}^*_{MU}(b) < \hat{R}_{MU}\}}{B}\right),$$

where Φ^{-1} is the inverse function of a standard normal cumulative distribution function. There are various ways to compute the acceleration $\hat{a}$. Usually $\hat{a}$ is calculated by jackknife method.

Let X^{-i} be the original sample with the ith point x_i deleted, let $\hat{R}_{MU}{}^{-i} = \hat{R}(X^{-i})$, and define $\hat{R}(\cdot)=\sum_{i=1}^{n}\hat{R}_{MU}{}^{-i}/n$. The acceleration is given by

$$\hat{a}=\frac{\sum_{i=1}^{n}(\hat{R}(\cdot)-\hat{R}_{MU}{}^{-i})^3}{6\left[\sum_{i=1}^{n}(\hat{R}(\cdot)-\hat{R}_{MU}{}^{-i})^2\right]^{3/2}}.$$

From (2.6), (2.7), (2.8), we calculate the bootstrap confidence intervals for R for sample sine n = 10(20)50 (based on 100 Monte Carlo runs and B=1,000) when the location parameter $\theta=1$ and the scale paremater σ=0.5(0.5)1.5, and time t=1.2, 2, and then obtain the average length (AL) of approximate confidence intervals

$$AL=\sum_{i=1}^{100}\frac{(\hat{R}_{\%,up}(i)-\hat{R}_{\%,lo}(i))}{100}$$

and the percentage of trials that the indicated interval missed the true value on the left (% Miss Left) or right (% Miss Right) side, and coverage probability (CP) of bootstrap confidence intervals. These values are given in Table 2.

From Table 2, we obtain the following results;

1. When the sample size is samll (n=10) and R is small, PI is better than BC_a in terms of AL but BC_a is better than PI in terms of CP.

2. When the sample size is large ($n \geq 30$), BC_a is better than PI in terms of AP and CP in the case of small R, but PI is better than BC_a in the case of large R. Since PI method is simple, we know that the bootstrap percentile interval provides good interval estimation unless n is large and R is large.

Table 1. The mean squared errors for the estimators of R.

n	Estimator	R=.37	R=.51	R=.67	R=.82	R=.88
10	$\hat{R}_{MLE}$	.01601	.01709	.01628	.01052	.00692
	$\hat{R}_{MU}$	.01350	.01362	.01233	.00763	.00531
	$\hat{R}_{MM}$	.01500	.01656	.01533	.00939	.00669
	$\hat{R}_{UU}$	.01330	.01353	.01235	.00779	.00561
30	$\hat{R}_{MLE}$	.00470	.00440	.00314	.00208	.00168
	$\hat{R}_{MU}$	.00441	.00409	.00282	.00159	.00107
	$\hat{R}_{MM}$	.00468	.00449	.00312	.00175	.00117
	$\hat{R}_{UU}$	.00441	.00409	.00282	.00159	.00108
50	$\hat{R}_{MLE}$	.00286	.00250	.00168	.00094	.00075
	$\hat{R}_{MU}$	.00275	.00239	.00159	.00077	.00053
	$\hat{R}_{MM}$	.00283	.00253	.00171	.00082	.00057
	$\hat{R}_{UU}$	.00275	.00239	.00159	.00077	.00054

Table 2. Comparision of bootstrap confidence intervals for R
($\sigma=1.0, \theta=1.0, t=2.0, R=.37$)

n	Method	AL	%Miss Left	%Miss Right	CP(90%)
10	PI	.35499	0	15	85
	STI	.39677	1	7	92
	BC_a	.37167	10	1	89
30	PI	.21454	3	14	83
	STI	.22161	5	8	87
	BC_a	.21347	11	3	86
50	PI	.16826	2	8	90
	STI	.17145	3	4	93
	BC_a	.16728	4	2	94

Table2.(continued) ($\sigma=1.5, \theta=1.0, t=2.0, R=.51$)

n	Method	AL	%Miss Left	%Miss Right	CP(90%)
10	PI	.37066	2	12	86
	STI	.41584	2	5	93
	BC_a	.38640	10	3	87
30	PI	.21089	1	9	90
	STI	.21871	3	3	94
	BC_a	.20509	5	3	92
50	PI	.16120	4	10	86
	STI	.16475	4	7	89
	BC_a	.15876	4	5	91

Table2.(continued) ($\sigma=0.5, \theta=1.0, t=1.2, R=.67$)

n	Method	AL	%Miss Left	%Miss Right	CP(90%)
10	PI	.34880	4	17	79
	STI	.39143	4	3	93
	BC_a	.37393	9	2	89
30	PI	.17535	1	12	87
	STI	.18286	4	5.	91
	BC_a	.17061	8	5	87
50	PI	.13031	3	2	95
	STI	.13317	4	1	95
	BC_a	.12670	6	1	93

Table2.(continued) ($\sigma=1.0, \theta=1.0, t=1.2, R=.82$)

n	Method	AL	%Miss Left	%Miss Right	CP(90%)
10	PI	.29492	10	2	88
	STI	.31416	9	0	91
	BC_a	.28089	11	0	89
30	PI	.13101	7	7	86
	STI	.13712	5	2	93
	BC_a	.13887	13	2	85
50	PI	.09098	4	7	89
	STI	.09379	4	2	94
	BC_a	.09313	9	2	89

Table2.(continued) ($\sigma=1.5, \theta=1.0, t=1.2, R=.88$)

n	Method	AL	%Miss Left	%Miss Right	CP(90%)
10	PI	.24576	16	3	81
	STI	.26520	16	0	84
	BC_a	.22160	17	0	83
30	PI	.11269	13	4	83
	STI	.11645	10	2	88
	BC_a	.12458	17	0	83
50	PI	.07663	5	3	92
	STI	.07951	5	3	92
	BC_a	.08240	7	2	91

References

[1] Abedin, Z. and Karson, M.(1993). Classical and Bayesian Estimation of Exponential Reliability under Time Censoring with Replacement, *Communications Statistics-Simulation*, Vol. 22(2), 471-496.

[2] Efron, B.(1979). Bootstrap Methods: Another Look at the Jackknife, *The Annals of Statistics,* Vol. 7, 1-26.

[3] Efron, B.(1981). Nonparametric Standard Errors and Confidence Interval, *Canadian Journal of Statistics,* Vol. 9, 139-172.

[4] Efron, B.(1982). Transformation Theory: How Normal is a Family of Distributions ?, *The Annals of Statistics,* Vol. 10, 323-339.

[5] Efron, B.(1985). Bootstrap Confidence Intervals for a Class of Parametric Problems, *Biometrika,* Vol. 72, 45-58.

[6] Efron, B.(1987). Better Bootstrap Confidence Intervals, *Journal of the American Statistical Association,* Vol. 82, 171-185.

[7] Efron, B. and Tishirani, J. R.(1993). *An Introduction to the Bootstrap*, Chapman & Hall.

[8] Epstein, B. and Sobel, M.(1954). Some Theorems Relebant to Life Testing from an Exponential Distribution, *Annals of Mathematical Statistcs*, Vol. 25, 373-381.

[9] Kang, S. B.(1987). Jackknife Estimation of the Reliability Function in an Exponential Distribution, *Journal of natural Sciences*, Yeungnam Uinversity, Vol. 7, 27-36.

[10] Kang, S. B. and Cho, Y. S.(1997). Bootstrap Confidence Intervals After Bias Correcting, *Far East Journal Theoretical Statistics,* Vol. 1, No. 1, 31-38.

[11] Shao, J. and Tu, D.(1995). *The Jackknife and Bootstrap*, Springer-Verlag, New York.

[12] Singh, H. N. and Gupta, R. D.(1993). Estimation of Parameters of an Expontial Distribution When the Parameter Space is Restricted with an Application to

Two-sample Problem, *Communication Statistics-Theory and Methods,* Vol. 22(2), 461-477.

[13] Sinha, S. K.(1985). Bayes Estimation of the Reliability Function of Normal Distribution, *IEEE Transactions of Reliability*, Vol. R-34, 360-362.

[14] Woo, J. S. and Kang, S. B.(1986). Jackknife Estimation of the Reliability Function in a Poisson Distribution, *Journal of natural Sciences*, Yeungnam Uinversity, Vol. 6, 29-34.