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Second Derivative Estimation for Performance Measures in a Markov Renewal Process

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Abstract

In this paper, we find the second derivative of mean busy cycle with respect to a parameter of inter-arrival time distribution. We show that this derivative can be estimated from a single sample path. We do the similar thing for the mean number of arrivals during busy cycle.

1. Introduction

Recently, perturbation analysis method has been introduced as an efficient way of estimating derivatives for performance measures from a single sample path of a discrete-event system. To estimate derivative with respect to a given parameter, the perturbation analysis method requires only one simulation run. But the traditional finite difference derivative estimates involve two or more simulation runs. The advantage in the number of simulation runs becomes significant when the differential parameter is an N-dimensional vector. Estimating the entries of the NxN Hessian matrix by means of finite differences would require $2N^2+1$ simulation runs whereas the perturbation analysis method still requires only one.

Since it was introduced, perturbation analysis method has been applied to many discrete event systems and proved to be an efficient way to find derivative estimation for the mean performance measures[Suri and Zazanis(1988), Glasserman and Gong(1990), Park(1992), etc.]. However, with the exception of [Fu and Hu(1993), Zazanis and Suri(1994)] all previous work on derivative estimation has focused on the problem of first derivative estimation. Zazanis and Suri(1994) considered the steady state mean system time in a GI/G/1 queue. They showed the second derivative with respect to a parameter of the service time distribution exists and obtained strongly consistent perturbation analysis estimates for it as well. They extended these results to a parameters of the interarrival time distribution. Fu and Hu(1993) considered the GI/G/m queue and derived an estimator for the second derivative of mean steady-state system time with respect to a parameter of the service time distribution. In this paper, we

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consider a Markov renewal process which was defined in Park(1996), and derive the second derivative of some performance measures with respect to a parameter of given distribution.

2. Second Derivative Estimation of the Mean Number of Arrivals During Busy Cycle

Let X and Y are independent random variables which are distributed according to F(x,θ) and G(y,μ) respectively. Using these X and Y, we construct the underlying stochastic process as in Park(1996), that is, the process will be a Markov renewal process where the embedded Markov chain is a random walk on the nonnegative integers with a reflecting barrier at zero. As the sample paths have an obvious interpretations as queueing process we will use freely terms such as "busy cycle", "arrivals", etc,. In this section, we will find the second derivative of mean number of arrivals during busy cycle with respect to the parameter θ of distribution function F(x,θ). We will use the same notations as in Park(1996). But for convenience we will define some notations here again.

$C_{0,0}(\theta)$ denotes the length of a busy cycle i.e. the recurrence time of state 0. $C_{i,0}(\theta)$ denotes the time for the first transition from state i to state 0.

$N_{0,0}(\theta)$ means the number of customers served during a busy cycle $C_{0,0}(\theta)$. Similarly, $N_{i,0}(\theta)$ means the number of customers served during $C_{i,0}(\theta)$.

Let f(x) and g(y) be the probability density functions of X and Y respectively, i.e. dF(x, θ)/dx = f(x) and dG(y,μ)/dy = g(y). When we emphasize that f(x) depends on θ, we will use f(x,θ) instead of f(x).

With these notations, we now begin to derive the second derivative of $N_{0,0}(\theta)$ with respect to θ. From theorem 1 of Park(1996), we have

$$\begin{aligned}\frac{dE[N_{0,0}(\theta)]}{d\theta} &= \frac{E[f(Y)Y]}{\theta P(Y<X)}E[N_{0,0}(\theta)]E[1-N_{2,0}(\theta)] \\ &= \frac{E[f(Y)Y]}{\theta P(Y<X)}E[N_{0,0}(\theta)]\{1-2E[N_{0,0}(\theta)]\} \qquad (1)\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{E[f(Y)Y]}{P(Y<X)} &= \int_0^\infty \frac{f(y)y}{1-F(y)}\,\frac{1-F(y)}{P(Y<X)}\,g(y)dy \\ &= \int_0^\infty \frac{f(y)y}{1-F(y)}\,g(y|Y<X)dy \\ &= E[\frac{f(Y)Y}{1-F(Y)}\,|Y<X], \qquad (2)\end{aligned}$$

From (1) and (2), we see that $\frac{dE[N_{0,0}(\theta)]}{d\theta}$ can be observed from a single sample path. By differentiating both side of the expression (1), we see that $\frac{d^2E[N_{0,0}(\theta)]}{d\theta^2}$ also can be obtained from sample path if $\frac{d\{E[f(Y)Y]/P(Y<X)\}}{d\theta}$ can be obtained.

In the following, we will show that $\frac{d\{E[f(Y)Y]/P(Y<X)\}}{d\theta}$ can be obtained from sample path. First, we need the following assumptions.

Let a and b are positive numbers such that a<b.

A1. $P(X<Y)<1/2$ for all θ in [a,b].
A2. $E[X]<\infty$, $E[Y]<\infty$.
A3. θ is a scale parameter of $F(x,\theta)$.
A4. $|dF(x,\theta)/d\theta|\le M$ for some $M>0$, and for all $x\ge0$ and all θ in [a,b].
A5. $|df(x,\theta)/d\theta|\le M$ for some $M>0$, and for all $x\ge0$ and all θ in [a,b].
A6. $|f(x,\theta)|\le M$ for some $M>0$, and for all $x\ge0$ and all θ in [a,b].
$|g(y)|\le M$ for some $M>0$, and for all $y\ge0$.

A1,A2 and A3 are same as in Park(1996). It can be shown that A4, and A5 holds for the wide range of distributions which have scale parameter. For example, in the case of Weibull distribution with $\beta>1$,

$$\left|\frac{dF(Y)}{d\theta}\right| = \left|-\left(\frac{Y}{\theta}\right)^{\beta}\left(\frac{\beta}{\theta}\right)\exp\left[-\left(\frac{Y}{\theta}\right)^{\beta}\right]\right|$$
$$\le \left(\frac{Y}{a}\right)^{\beta}\left(\frac{\beta}{a}\right)\exp\left[-\left(\frac{Y}{b}\right)^{\beta}\right]$$

Since $\lim_{Y\to\infty}\left(\frac{Y}{a}\right)^{\beta}\left(\frac{\beta}{a}\right)\exp\left[-\left(\frac{Y}{b}\right)^{\beta}\right]=0$, there exists $M>0$ such that $|dF(Y)/d\theta|\le M$ for all $Y\ge0$ and all θ in [a,b] i.e. assumption A4 holds. Similarly, we can show that assumption A5 holds.

Now we prove the following theorem.

Theorem 1. Assume that $\frac{dE[N_{0,0}(\theta)]}{d\theta}$ exists and assumptions A1 through A6 hold. Then, the second derivative $\frac{d^2E[N_{0,0}(\theta)]}{d\theta^2}$ also exists and it can be estimated from a single sample path.

Proof) From the arguments following expression (2), it will be enough to show that $\frac{d\{E[f(Y)Y]/P(Y<X)\}}{d\theta}$ can be obtained from a single sample path.

$$\frac{dE[f(Y)Y]}{d\theta} = \lim_{\Delta\theta \to 0} \frac{E[f(Y,\theta)Y - f(Y,\theta-\Delta\theta)Y]}{\Delta\theta}$$

$$= \lim_{\Delta\theta \to 0} E[\frac{df(Y,\theta)Y}{d\theta} \mid_{\theta=\xi}], \tag{3}$$

where ξ is a number between θ and $\theta-\Delta\theta$.

By the assumptions A2 and A5, we can use Lebesgue convergence theorem on the above expression (3). Then the above expression (3) becomes

$$E[\lim_{\Delta\theta \to 0} \frac{df(Y,\theta)Y}{d\theta} \mid_{\theta=\xi}]$$

$$= E[\frac{df(Y)Y}{d\theta}] \tag{4}$$

Since $P(Y<X)=E[1-F(Y)]$,

$$\frac{dP(Y<X)}{d\theta} = \lim_{\Delta\theta \to 0} \frac{E[F(Y,\theta)-F(Y,\theta+\Delta\theta)]}{\Delta\theta}$$

$$= -\lim_{\Delta\theta \to 0} E[\frac{dF(Y,\theta)}{d\theta} \mid_{\theta=\xi}], \tag{5}$$

where ξ is a number between θ and $\theta+\Delta\theta$. By the assumption A4 and Lebesgue convergence theorem, the above expression (5) becomes

$$-E[\lim_{\Delta\theta \to 0} \frac{dF(Y,\theta)}{d\theta} \mid_{\theta=\xi}]$$

$$= -E[\frac{dF(Y)}{d\theta}] \tag{6}$$

From the expressions (3), (4), (5), and (6),

$$\frac{d\{E[f(Y)Y]/P(Y<X)\}}{d\theta} = \frac{E[df(Y)Y/d\theta]}{P(Y<X)} + \frac{E[f(Y)Y]}{P(Y<X)} \frac{E[dF(Y)/d\theta]}{P(Y<X)} \tag{7}$$

Using similar method as in the expression (2), the above expression (7) becomes

$$E[\frac{df(Y)Y/d\theta}{1-F(Y)} \mid Y<X] + E[\frac{f(Y)Y}{1-F(Y)} \mid Y<X]E[\frac{dF(Y)/d\theta}{1-F(Y)} \mid Y<X] \tag{8}$$

Every terms in the expression (8) can be observed from sample path. Hence, theorem 1 is proved.

3. Second Derivative Estimation for the Mean Length of Busy Cycle

In this section, we will show that the second derivative $\frac{d^2E[C_{0,0}(\theta)]}{d\theta^2}$ exists and it can be observed from sample path. First, let us define two more notations.

Let U be the set of indices at which the process jumps up during $C_{0,0}(\theta)$. |U| denotes the cardinal number during a busy cycle $C_{0,0}(\theta)$. We note that |U| equals to $N_{0,0}(\theta)$.

Theorem 2. Assume two more assumptions on those of theorem 1, namely

1. $E[X^2(\theta)]<M$ for all θ in [a,b] and some M>0
2. $\frac{dE[C_{0,0}(\theta)]}{d\theta}$ exists

Then, $\frac{d^2E[C_{0,0}(\theta)]}{d\theta^2}$ also exists and it can be estimated from a single sample path.

Proof) From theorem 2 in Park(1996), we have

$$\frac{dE[C_{0,0}(\theta)]}{d\theta}=\frac{1}{\theta}E[\sum_{j\in U}X_j]-\frac{E[f(Y)Y]}{\theta P(Y<X)}E[N_{0,0}(\theta)]E[C_{2,0}(\theta)] \tag{9}$$

On the other hand,

$$E[C_{2,0}(\theta)]=2E[C_{1,0}(\theta)]=2\{E[C_{0,0}(\theta)]-E[X_1]\}$$

and

$$\begin{aligned}E[\sum_{j\in U}X_j]&=E[X_1]+E[X|X<Y]E[|U|-1]\\&=E[X_1]+E[X|X<Y]E[N_{0,0}(\theta)-1]\end{aligned}$$

Hence, the above expression (9) becomes

$$\begin{aligned}&\frac{1}{\theta}\{E[X_1]+E[X|X<Y]E[N_{0,0}(\theta)-1]\}\\&\qquad-\frac{2}{\theta}\frac{E[f(Y)Y]}{P(Y<X)}E[N_{0,0}(\theta)]\{E[C_{0,0}(\theta)]-E[X_1]\}\end{aligned} \tag{10}$$

We know that every terms in the above expression (10) can be observed from sample path (for $\frac{E[f(Y)Y]}{P(Y<X)}$, see the expression (2)). Since θ is a scale parameter, $E[X_1]$ is differentiable and

$$\frac{dE[X_1]}{d\theta}=E[\frac{dX_1}{d\theta}]=\frac{1}{\theta}E[X_1]\ \text{[see Suri and Zazanis(1988)]} \tag{11}$$

Hence, from theorem 1 and assumptions we also see that every terms except E[X|X<Y] in the expression (10) are differentiable and the derivatives can be obtained from sample path. Consequently, if we show that E[X|X<Y] is differentiable with respect to θ and $\frac{dE[X|X<Y]}{d\theta}$ can be observed from sample path, then the theorem 2 will be proved.

We now show that E[X|X<Y] is differentiable with respect to θ.

$$\begin{aligned} E[X|X<Y] &= \int_0^\infty x f(x|X<Y)dx \\ &= \int_0^\infty x\frac{1-G(x)}{P(X<Y)}f(x)dx \\ &= \frac{E[X\{1-G(X)\}]}{P(X<Y)} \\ &= \frac{E[X]-E[XG(X)]}{P(X<Y)} \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned} \frac{dE[XG(X)]}{d\theta} &= \lim_{\Delta\theta\to 0}\frac{E[X(\theta)G(X(\theta))-X(\theta-\Delta\theta)G(X(\theta-\Delta\theta))]}{\Delta\theta} \\ &= \lim_{\Delta\theta\to 0}E[\frac{X(\xi)}{\xi}G(X(\xi))+\frac{X^2(\xi)}{\xi}g(X(\xi))], \end{aligned} \tag{13}$$

where ξ is a number between θ-Δθ and θ. Since, for all ξ in [a,b],

$$\frac{X(\xi)}{\xi}G(X(\xi))+\frac{X^2(\xi)}{\xi}g(X(\xi))\le\frac{X(b)}{a}+\frac{X^2(b)}{a}M$$

and

$$E[X^2(b)]<\infty,$$

by the Lebesgue convergence theorem the above expression (13) becomes

$$\begin{aligned} &E[\lim_{\Delta\theta\to 0}\{\frac{X(\xi)}{\xi}G(X(\xi))+\frac{X^2(\xi)}{\xi}g(X(\xi))\}] \\ &= E[\frac{X}{\theta}G(X)+\frac{X^2}{\theta}g(X)]<\infty \end{aligned} \tag{14}$$

Similarly as in the expressions (5) and (6),

$$\frac{dP(X<Y)}{d\theta}=\frac{dE[1-G(X)]}{d\theta}=-E[g(X)\frac{X}{\theta}] \tag{15}$$

From the expressions (11) through (15), we see that E[X|X<Y] is differentiable with respect to θ and the derivative $\frac{dE[X|X<Y]}{d\theta}$ is given by the following formula.

$$\frac{dE[X|X<Y]}{d\theta} = \frac{dE[X]/d\theta}{P(X<Y)} - \frac{E[X]}{P(X<Y)}\frac{dP(X<Y)/d\theta}{P(X<Y)}$$

$$-\frac{dE[XG(X)]/d\theta}{P(X<Y)} + \frac{E[XG(X)]}{P(X<Y)}\frac{dP(X<Y)/d\theta}{P(X<Y)}$$

$$= \frac{1}{\theta}\frac{E[X]}{P(X<Y)} + \frac{1}{\theta}\frac{E[X]}{P(X<Y)}\frac{E[Xg(X)]}{P(X<Y)}$$

$$-\frac{1}{\theta}\frac{E[XG(X)]}{P(X<Y)} - \frac{1}{\theta}\frac{E[X^2g(X)]}{P(X<Y)} - \frac{1}{\theta}\frac{E[XG(X)]}{P(X<Y)}\frac{E[Xg(X)]}{P(X<Y)}$$

$$= \frac{1}{\theta}\{\frac{E[X]}{P(X<Y)} - \frac{E[XG(X)]}{P(X<Y)}\}\{1 + \frac{E[Xg(X)]}{P(X<Y)}\} - \frac{1}{\theta}\frac{E[X^2g(X)]}{P(X<Y)}$$

$$= \frac{1}{\theta}E[X|X<Y]\{1 + \frac{E[Xg(X)]}{P(X<Y)}\} - \frac{1}{\theta}\frac{E[X^2g(X)]}{P(X<Y)} \tag{16}$$

Similarly as in (2), the above expression (16) becomes

$$\frac{1}{\theta}E[X|X<Y]\{1 + E[\frac{Xg(X)}{1-G(X)}|X<Y]\} - \frac{1}{\theta}E[\frac{X^2g(X)}{1-G(X)}|X<Y] \tag{17}$$

If the underlying process jumps up at the end of the i-th sojourn time and the i-th state is greater than zero, then $X_i < Y_i$. Since those sojourn times X_is are i.i.d., we can estimate all the terms of the expression (17) by observing the sample path. Hence the theorem 2 is proved.

4. Concluding Remarks

In this paper, we showed that, under the moderate assumptions, the second derivative of mean busy cycle with respect to a scale parameter θ of inter-arrival time distribution exists and it can be estimated from a single sample path. We also showed similar thing for the mean number of arrivals during a busy cycle. As we can see from the proofs of theorems in this paper, these results will be easily extended to the third or higher derivatives.

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