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Unbiased Estimators of Standard Deviation in a Truncated Arcsine Distribution[1)]

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Abstract

Three kinds of unbiased estimators of standard deviation in a truncated arcsine distribution based on the quasi-range, the jackknife quasi-range, and the angle jackknife range are proposed and numerically compared each other in a sense of MSE.

1. Introduction

Let T ~ Tarcsin(β) be a truncated arcsine distribution with probability density function(pdf)

$$f(x)=\frac{2}{\pi}\frac{1}{\sqrt{\beta^2-x^2}}\ ,\ 0<x<\beta\ .$$

Then the mean and variance of T are $\frac{2}{\pi}\beta$ and $(\frac{1}{2}-\frac{4}{\pi^2})\beta^2$, respectively. The special standard bete distribution with parameters $\frac{1}{2}$ (as known an arcsine distribution) arises in an interesting way in the theory of random walks(see Johnson et al(1995)).

Let $T_1, T_2, \cdots, T_n$ be a simple random sample(SRS) from a truncated arcsine distribution with a scale parameter β and let $T_{(1)} \le T_{(2)} \le \cdots \le T_{(n)}$ be the corresponding order statistics.

Result 1.1 (Woo(1996)) Let T= $\beta\cos\theta$.

T ~ Tarcsin(β) iff θ follows a uniform distribution over $(0, \pi/2)$.

Let $C(n; a, b)=\int_0^{\frac{\pi}{2}} x^n\cos(ax+b)dx$ and $S(n; a, b)=\int_0^{\frac{\pi}{2}} x^n\sin(ax+b)dx$, where $a \neq 0$ and n is a non-negative integer.

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From formulas 2.633(1) & (2) in Gradshteyn et al (1965), we can obtain the following :

Result 1.2

a. $$C(n;a,b)=(\frac{1}{a})^{n+1}\sum_{r=0}^{n}\sum_{s=0}^{r}(-1)^{n-r}s!\binom{n}{r}\binom{r}{s}[b^{n-r}(\frac{\pi}{2}a+b)^{r-s}\cdot\sin(\frac{\pi}{2}(a+s)+b)-b^{n-s}\sin(\frac{\pi}{2}s+b)].$$

b. $$S(n;a,b)=(\frac{1}{a})^{n+1}\sum_{r=0}^{n}\sum_{s=0}^{r}(-1)^{n-r+1}s!\binom{n}{r}\binom{r}{s}[b^{n-r}(\frac{\pi}{2}a+b)^{r-s}\cdot\cos(\frac{\pi}{2}(a+s)+b)-b^{n-s}\cos(\frac{\pi}{2}s+b)].$$

Let $\theta_1,\theta_2,\cdots,\theta_n$ be a SRS from a uniform distribution over $(0,\pi/2)$ and let $\theta_{(1)}\le\theta_{(2)}\le\cdots\le\theta_{(n)}$ be the corresponding order statistics. From Result 1.2 and the probability density function(pdf) of $\theta_{(i)}$, $i=1,\cdots,n$, we can obtain the following expectations :

Result 1.3 For $i=1,\cdots,n$,

a. $$E_i(1)\equiv E(\cos\theta_{(i)})=i\cdot\binom{n}{i}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}(\frac{2}{\pi})^{i+k}\cdot C(i+k-1;1,0).$$

b. $$E_i(2)\equiv E(\cos^2\theta_{(i)})=\frac{1}{2}+\frac{i}{2}\binom{n}{i}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}(\frac{2}{\pi})^{i+k}\cdot C(i+k-1;2,0).$$

From Result 1.2 and the joint pdf of $\theta_{(i)}$ and $\theta_{(j)}$, $1\le i\le j\le n$, we can obtain the following expectation :

Result 1.4 For $1\le i\le j\le n$,

$$E_{i,j}(1)=E(\cos\theta_{(i)}\cdot\cos\theta_{(j)})$$
$$=\frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{k=0}^{j-i-1}\sum_{p=0}^{i+k-1}\sum_{q=0}^{n-j}(-1)^k\binom{j-i-1}{k}p!\binom{i+k-1}{p}(-1)^q\binom{n-j}{q}\cdot$$
$$(\frac{\pi}{2})^{n-j-q}\cdot[\frac{1}{2}S(j+q-2;2,\frac{\pi}{2}p)+\frac{1}{2(j+q-1)}(\frac{\pi}{2})^{j+q-1}\sin\frac{\pi}{2}p$$
$$-(i+k-1)!\sin(\frac{\pi}{2}(i+k-1))\cdot C(j+q-i-k-1;1,0)].$$

2. Unbiased Estimators of Standard Deviation

Let $R_j=T_{(n-j+1)}-T_{(j)}$ be the quasi-range of the samples, j=1,2,...,[n/2](David(1981)), where [x] is the greatest integer not exceeding x. Then, from Results 1.1, 1.3, and 1.4, we can get the mean and variance of the quasi-range as follows :

Result 2.1 For j=1, 2, ... ,[n/2],

a. $E(R_j)=\beta(E_j(1)-E_{n-j+1}(1))$.

b. $Var(R_j)=\beta^2[E_j(2)+E_{n-j+1}(2)+2E_j(1)E_{n-j+1}(1)-E_j^2(1)-E_{n-j+1}^2(1)-2E_{j,n-j+1}(1)]$.

From the jackknife technique in Gray et al (1972), the ordinary jackknife estimator of the quasi-range R_j, j=1, 2, ... ,[n/2]-1, is

$$J(R_j)=\beta[\frac{2n-4j+3}{n-2j+2}(\cos\theta_{(j)}-\cos\theta_{(n-j+1)})-\frac{n-2j+1}{n-2j+2}(\cos\theta_{(j+1)}-\cos\theta_{(n-j)})].$$

From Results 1.3 and 1.4, we obtain the mean and variance of the ordinary jackknife estimator of the quasi-range as follows :

Result 2.2 For j=1, 2, ... ,[n/2]-1,

a. $E[J(R_j)]=\beta[\frac{2n-4j+3}{n-2j+2}(E_j(1)-E_{n-j+1}(1))-\frac{n-2j+1}{n-2j+2}(E_{j+1}(1)-E_{n-j}(1))]$.

b.
$$\begin{aligned} Var[J(R_j)]=\beta^2[&(\frac{2n-4j+3}{n-2j+2})^2(E_j(2)+E_{n-j+1}(2)-E_j^2(1)-E_{n-j+1}^2(1)-2E_{j,n-j+1}(1)\\ &\qquad +2E_j(1)\cdot E_{n-j+1}(1))\\ &+(\frac{n-2j+1}{n-2j+2})^2(E_{j+1}(2)+E_{n-j}(2)-E_{j+1}^2(1)-E_{n-j}^2(1)-2E_{j+1,n-j}(1)\\ &\qquad +2E_{j+1}(1)\cdot E_{n-j}(1))\\ &-2\frac{(2n-4j+3)(n-2j+1)}{(n-2j+2)^2}(E_{j,j+1}(1)-E_j(1)\cdot E_{j+1}(1)\\ &\quad +E_j(1)\cdot E_{n-j}(1)-E_{j,n-j}(1)+E_{j+1}(1)\cdot E_{n-j+1}(1)\\ &\quad -E_{j+1,n-j+1}(1)+E_{n-j,n-j+1}(1)-E_{n-j}(1)\cdot E_{n-j+1}(1)].\end{aligned}$$

Next we shall consider the angle jackknife estimator of the range in a truncated arcsine distribution with a scale parameter β. Let η_i be the ordinary jackknife of $\theta_{(i)}$, $i=1,\cdots,n$. From the jackknife technique in Gray et al(1972), we can obtain the jackknife estimators of $\theta_{(i)}$, i=1 and n as follows :

$$\eta_1=\frac{2n-1}{n}\theta_{(1)}-\frac{n-1}{n}\theta_{(2)} \text{ and } \eta_2=\frac{2n-1}{n}\theta_{(n)}-\frac{n-1}{n}\theta_{(n-1)}.$$

Therefore, the angle jackknife estimator $J_a(R_1)$ of the range, R_1, can be defined as

$$J_a(R_1)=\beta(\cos\eta_1-\cos\eta_2).$$

The pdf's of η_i, i=1 and 2 can be obtained as follows :

$$f_{\eta_1}(x)=\begin{cases}\frac{n^2}{2n-1}(\frac{2}{\pi})^n(\frac{\pi}{2}-x)^{n-1}, & 0<x<\frac{\pi}{2}\\ \frac{n^2}{2n-1}(\frac{2}{\pi})^n(\frac{\pi}{2}+\frac{n}{n-1}x)^{n-1}, & -\frac{n-1}{n}\cdot\frac{\pi}{2}<x<0\end{cases}$$

and $$f_{\eta_2}(x)=\begin{cases}\frac{n^2}{2n-1}(\frac{2}{\pi})^n x^{n-1}, & 0<x<\frac{\pi}{2}\\ \frac{n^2}{2n-1}(\frac{n}{n-1})^{n-1}(\frac{2}{\pi})^n(\frac{2n-1}{n}\frac{\pi}{2}-x)^{n-1}, & \frac{\pi}{2}<x<\frac{2n-1}{n}\frac{\pi}{2}.\end{cases}$$

From Result 1.2 and the pdf's of η_i , i=1 and 2, we can obtain the following expectations :

Result 2.3

a. $E(\cos\eta_1))=\frac{n^2}{2n-1}(\frac{2}{\pi})^n S(n-1;1,0)+\frac{n}{2n-1}(\frac{2}{\pi})^n(n-1)C(n-1;\frac{n-1}{n},-\frac{n-1}{n}\frac{\pi}{2}).$

b. $$E(\cos^2\eta_1)=\frac{1}{2}\frac{n^2}{2n-1}(\frac{2}{\pi})^n[\frac{1}{n}(\frac{\pi}{2})^n+\frac{n-1}{n^2}(\frac{\pi}{2})^n+C(n-1;2,0)+\frac{n-1}{n}C(n-1;2\frac{(n-1)}{n},-\frac{n-1}{n}\pi)].$$

c. $$E(\cos\eta_2)=\frac{n^2}{2n-1}(\frac{2}{\pi})^n C(n-1;1,0)+\frac{n^2}{2n-1}(\frac{n}{n-1})^{n-1}(\frac{2}{\pi})^n\cdot[\cos(\frac{2n-1}{n}\frac{\pi}{2})\cdot(\sum_{k=0}^{n-1}k!\binom{n-1}{k}(\frac{n-1}{n}\frac{\pi}{2})^{n-k-1}\sin(\frac{\pi}{2}(\frac{n-1}{n}+k))-(n-1)!\sin(\frac{n-1}{2}\pi))+\sin(\frac{2n-1}{n}\frac{\pi}{2})((n-1)!\cos(\frac{n-1}{2}\pi)-\sum_{k=0}^{n-1}k!\binom{n-1}{k}(\frac{n-1}{n}\frac{\pi}{2})^{n-k-1}\cdot\cos(\frac{\pi}{2}(\frac{n-1}{n}+k)))].$$

d. $$E(\cos^2\eta_2)=\frac{n}{2(2n-1)}+\frac{n^2}{2(2n-1)}(\frac{2}{\pi})^n C(n-1:2,0)+\frac{n-1}{2(2n-1)}(\frac{n}{n-1})^{n-1}(\frac{2}{\pi})^{n-1}+\frac{n^2}{2(2n-1)}(\frac{n}{n-1})^{n-1}(\frac{2}{\pi})^n\cos(\frac{2n-1}{n}\pi)[\sum_{k=0}^{n-1}k!\binom{n-1}{k}(\frac{1}{2})^{k+1}\cdot(\frac{n-1}{n}\frac{\pi}{2})^{n-k-1}\sin(\frac{n-1}{n}\pi+\frac{k}{2}\pi)-(n-1)!(\frac{1}{2})^n\sin(\frac{n-1}{2}\pi)]+\frac{n^2}{2(2n-1)}(\frac{n}{n-1})^{n-1}(\frac{2}{\pi})^n\sin(\frac{2n-1}{n}\pi)[-\sum_{k=0}^{n-1}k!\binom{n-1}{k}(\frac{1}{2})^{k+1}\cdot(\frac{n-1}{n}\frac{\pi}{2})^{n-k-1}\cos(\frac{n-1}{n}\pi+\frac{k}{2}\pi)+(n-1)!(\frac{1}{2})^n\cos(\frac{n-1}{2}\pi)].$$

From Result 1.2 and the joint pdf of $\theta_{(1)}, \theta_{(2)}, \theta_{(n-1)}$, and $\theta_{(n)}$, we can obtain an expectation of $\cos\eta_1\cdot\cos\eta_2$. It has been difficult to induce the integrals enabling us to find the expectation.

Result 2.4 Let c=n(n-1)(n-2)(n-3) $(\frac{2}{\pi})^n$. Then

$$E(\cos\eta_1\cdot\cos\eta_2)=c\cdot(\frac{n}{2n-1})^2\sin(\frac{2n-1}{n}\frac{\pi}{2})\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}(\frac{n}{n-1})^{p+1}\cdot$$
$$[\frac{1}{2}(\frac{\pi}{2})^{k-p}\sin((\frac{n-1}{n}+p)\frac{\pi}{2})\cdot(S(n-k-4;\frac{3n-2}{n},0)+S(n-k-4;1,0))$$
$$-\frac{1}{4}(C(n-p-4;\frac{2n-1}{n},-\frac{p}{2}\pi)-C(n-p-4;\frac{4n-3}{n},\frac{p}{2}\pi)$$
$$+C(n-p-4;\frac{1}{n},-\frac{p}{2}\pi)-C(n-p-4;\frac{2n-1}{n},\frac{p}{2}\pi)\ )\ ]$$
$$+c\cdot(\frac{n}{2n-1})^2\cos(\frac{2n-1}{n}\frac{\pi}{2})\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}(\frac{n}{n-1})^{p+1}\cdot$$
$$[\frac{1}{2}(\frac{\pi}{2})^{k-p}\cos((\frac{n-1}{n}+p)\frac{\pi}{2})\cdot(S(n-k-4;\frac{3n-2}{n},0)+S(n-k-4;1,0))$$
$$-\frac{1}{4}(S(n-p-4;\frac{4n-2}{n},\frac{p}{2}\pi)+S(n-p-4;\frac{2n-1}{n},-\frac{p}{2}\pi)$$
$$+S(n-p-4;\frac{2n-1}{n},\frac{p}{2}\pi)+S(n-p-4;\frac{1}{n},-\frac{p}{2}\pi)\ )\ ]$$
$$+c\cdot(\frac{n}{2n-1})^2\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}\cdot$$
$$[\frac{1}{2}(\frac{\pi}{2})^{k-p}\cos((1+p)\frac{\pi}{2})\cdot(S(n-k-4;\frac{3n-2}{n},0)+S(n-k-4;1,0))$$
$$-\frac{1}{4}(S(n-p-4;\frac{4n-2}{n},\frac{p}{2}\pi)+S(n-p-4;\frac{2n-2}{n},-\frac{p}{2}\pi)$$
$$+S(n-p-4;2,\frac{p}{2}\pi)-\frac{1}{n-p-3}(\frac{\pi}{2})^{n-p-3}\sin(\frac{p}{2}\pi)\ )\ ]$$
$$+\ c\cdot(\frac{n}{2n-1})^2\sin(\frac{2n-1}{n}\frac{\pi}{2})\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}(\frac{n}{n-1})^{p+1}\cdot$$
$$[(\frac{\pi}{2})^{k-p}\sin((\frac{n-1}{n}+p)\frac{\pi}{2})\cdot S(n-k-4;\frac{n-1}{n},0)$$
$$+\frac{1}{2}C(n-p-4;\frac{2n-2}{n},\frac{p}{2}\pi)-\frac{1}{2(n-p-3)}(\frac{\pi}{2})^{n-p-3}\cos(\frac{p}{2}\pi)\ )$$
$$+C(n-p-4;\frac{1}{n},-\frac{p}{2}\pi)-C(n-p-4;\frac{2n-1}{n},\frac{p}{2}\pi)\ )\ ]$$
$$+c\cdot(\frac{n}{2n-1})^2\cos(\frac{2n-1}{n}\frac{\pi}{2})\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}(\frac{n}{n-1})^{p+1}\cdot$$
$$[(\frac{\pi}{2})^{k-p}\cos((\frac{n-1}{n}+p)\frac{\pi}{2})\cdot S(n-k-4;\frac{n-1}{n},0)$$
$$-\frac{1}{2}S(n-p-4;\frac{2n-2}{n},\frac{p}{2}\pi)+\frac{1}{2(n-p-3)}(\frac{\pi}{2})^{n-p-3}\sin(\frac{p}{2}\pi)\ ]$$

$$+\ c\cdot\left(\frac{n}{2n-1}\right)^2\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}\cdot$$

$$\Big[\left(\frac{\pi}{2}\right)^{k-p}\cos\left((1+p)\frac{\pi}{2}\right)\cdot S\left(n-k-4;\frac{n-1}{n},0\right)$$

$$+\frac{1}{2}S\left(n-p-4;\frac{1}{n},\frac{p}{2}\pi\right)-\frac{1}{2}S\left(n-p-4;\frac{2n-1}{n},\frac{p}{2}\pi\right)\Big]$$

$$+\ c\cdot\frac{1}{2}\left(\frac{n}{2n-1}\right)^2\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}\cdot$$

$$\Big[-\left(\frac{\pi}{2}\right)^{k-p}\cos\left((1+p)\frac{\pi}{2}\right)\cdot\left(S\left(n-k-4;\frac{3n-2}{n},0\right)-S(n-k-4;1,0)\right)$$

$$+\frac{1}{2}S\left(n-p-4;\frac{4n-2}{n},\frac{p}{2}\pi\right)+\frac{1}{2}S\left(n-p-4;\frac{2n-2}{n},-\frac{p}{2}\pi\right)$$

$$-\frac{1}{2}S\left(n-p-4;2,\frac{p}{2}\pi\right)+\frac{1}{2(n-p-3)}\left(\frac{\pi}{2}\right)^{n-p-3}\sin\left(\frac{p}{2}\pi\right)\Big]$$

$$+\ c\cdot\frac{1}{2}\cdot\left(\frac{n}{2n-1}\right)^2\sin\left(\frac{2n-1}{n}\frac{\pi}{2}\right)\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}\left(\frac{n}{n-1}\right)^{p+1}\cdot$$

$$\Big[-\left(\frac{\pi}{2}\right)^{k-p}\sin\left(\left(\frac{n-1}{n}+p\right)\frac{\pi}{2}\right)\cdot\left(S\left(n-k-4;\frac{3n-2}{n},0\right)-S(n-k-4;1,0)\right)$$

$$+\frac{1}{2}\Big(C\left(n-p-4;\frac{2n-1}{n},-\frac{p}{2}\pi\right)-C\left(n-p-4;\frac{4n-3}{n},\frac{p}{2}\pi\right)$$

$$-C\left(n-p-4;\frac{1}{n},-\frac{p}{2}\pi\right)+C\left(n-p-4;\frac{2n-1}{n},\frac{p}{2}\pi\right)\Big)\Big]$$

$$+\ c\cdot\frac{1}{2}\left(\frac{n}{2n-1}\right)^2\cos\left(\frac{2n-1}{n}\frac{\pi}{2}\right)\sum_{k=0}^{n-4}\sum_{p=0}^{k}(-1)^{n-k-4}p!\binom{n-4}{k}\binom{k}{p}\left(\frac{n}{n-1}\right)^{p+1}\cdot$$

$$\Big[-\left(\frac{\pi}{2}\right)^{k-p}\cos\left(\left(\frac{n-1}{n}+p\right)\frac{\pi}{2}\right)\cdot\left(S\left(n-k-4;\frac{3n-2}{n},0\right)-S(n-k-4;1,0)\right)$$

$$+\frac{1}{2}\Big(S\left(n-p-4;\frac{4n-3}{n},\frac{p}{2}\pi\right)+S\left(n-p-4;\frac{2n-1}{n},-\frac{p}{2}\pi\right)$$

$$-S\left(n-p-4;\frac{2n-1}{n},\frac{p}{2}\pi\right)-S\left(n-p-4;\frac{1}{n},-\frac{p}{2}\pi\right)\Big].$$

Proposed unbiased estimators of standard deviation σ in the truncated arcsine distribution can be given by $\widehat{\sigma_R}=\dfrac{R_j}{d_{R_j}}$, $\widehat{\sigma_J}=\dfrac{J(R_j)}{d_{J(R_j)}}$, and $\widehat{\sigma_A}=\dfrac{J_a(R_1)}{d_{J_a}}$,

where $d_{R_j}=\dfrac{E(R_j)}{\sigma}$, $d_{J(R_j)}=\dfrac{E(J(R_j))}{\sigma}$, and $d_{J_a}=\dfrac{E(J_a(R_1))}{\sigma}$ (See David(1981)).

3. Numerical Comparision of Variances and an Example

From Result 2.1 through 2.4, we can get the numerical values of variances of unbiased estimators $\widehat{\sigma_R}$, $\widehat{\sigma_J}$, and $\widehat{\sigma_A}$ of standard deviation in the truncated arcsine distribution when the sample size equals 10(20), β=1(and hence σ=0.30776).

Table. Variances of $\widehat{\sigma_R}$, $\widehat{\sigma_J}$, and $\widehat{\sigma_A}$ in the truncated arcsine distribution with a scale parameter β=1 and standard deviation σ=0.30776).

	n		10				20		
	j	1	2	3	1	2	3	4	5
	$\widehat{\sigma_R}$	0.0020	0.0054	0.0012	0.00045	0.00125	0.00211	0.00322	0.00470
Variance	$\widehat{\sigma_j}$	0.0041	0.0108	0.0220	0.00111	0.00497	0.00670	0.00911	0.01266
	$\widehat{\sigma_A}$	0.0019			0.00041				

The numerical results in Table show that variance of an unbiased estimator $\widehat{\sigma_A}$ of standard deviation is smaller than these of other two unbiased estimators. Especially variance of an unbiased estimator $\widehat{\sigma_R}$ is smaller than that of an unbiased estimator $\widehat{\sigma_j}$ of standard deviation in a truncated arcsine distribution. To estimate standard deviation based on the range in a truncated arcsine distribution we could more recommend the angle jackknifing estimator of the range than other two proposed estimators.

Example 3.1 (Nayar et al(1995)) <Lambert's Law>
Let ρ be the fraction of incident light reflected from the surface and θ be angle of incidence between surface normal and illumination direction. Then the brightness of the surface is

$$B=\frac{\rho}{\pi}\cos\theta\ ,\quad 0<\theta<\pi/2 \text{ and } \pi \text{ is } 3.14.$$

(a). If θ follows a uniform distribution over $(0,\pi/2)$, then B follows a truncated arcsine distribution with a known scale parameter ρ/π.

(b). When the sample size equals 20, then the minimum and maximum brightness are $0.9946\cdot\frac{\rho}{\pi}$ and $0.0744\cdot\frac{\rho}{\pi}$, respectively.

From the table, an unbiased estimator $\widehat{\sigma_A}$ of standard deviation in the truncated arcsine distribution based on the angle jackknife range could be recommended.

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