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Some Nonparametric Tests for Change-points with Epidemic Alternatives[1)]

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Abstract

The purpose of this paper is to discuss distribution-free tests of hypothesis that the random samples are identically distributed against the epidemic alternative. But most tests that have been considered are depended only on specific null distribution. Two nonparametric tests are considered and compared with a likelihood ratio test by the empirical powers.

1. Introduction

Let $\{ X_i , i=1,2,\ldots,n \}$ be the set of sequence of independent random variables from continuous distribution function $F(x, \theta_i)$, where θ is a unknown location parameter. For some integers $1 \le p < q \le n$, $X_1, \ldots, X_p, X_{q+1}, \ldots, X_n$ are identically distributed while $X_{p+1}, \ldots, X_q$ are also identically distributed. But the distributions of two sets of random variables are not equal. In this case p and q are called change-points. This kind of alternative is called epidemic alternative which is formulated by Levin & Kline(1985). Then the null hypothesis that has no change-points versus epidemic alternatives can be discribed more formally as

$$\begin{aligned} H_o &: \xi_1 = \xi_2 \\ H_1 &: \xi_1 \neq \xi_2, \end{aligned} \tag{1}$$

where the nuisance parameters ξ_1 and ξ_2 satisfy the equation such that $\theta_i = \xi_1 I(1 \le i \le p \ , \ q < i \le n) + \xi_2 I(p < i \le q)$ with indicator function $I(.)$.

Page(1954) first found methods based on cumulative sums for detecting change in distribution of sequence of random variables. But next decades there was a burst of activity which was based on parametric statistical procedures. During the 1980s, there were some

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articles which were interested in different types of change, eg. abrupt, smooth and epidemic changes. Lombard(1987) proposed nonparametric test procedure for testing multiple change-points. Recently epidemic structual change model have been used in economics. Broemeling & Tsurumi(1987) described a number of applications of this model in economics. Yao(1993a, 1993b) compared various kinds of tests and developed the approximations for the significance levels by using the boundary-crossing probability under normal null distribution.

In this paper one parametric likelihood ratio test statistic and the other two nonparametric statistics are introduced in section 2. In section 3, empirical powers for epidemic alternatives are reported to compare the different test procedures.

2. Test statistics

2.1 Likelihood ratio statistic

Under normal assumption, Sigmund(1988) proposed likelihood ratio test statistic for epidemic alternatives as follows :

$$S_n = \max_{m_0 \le q-p \le m_1} \left(S_q - S_p - \frac{q-p}{n} S_n \right) / \left\{ (q-p)\left(1 - \frac{q-p}{n}\right) \right\}^{1/2}$$

for some $1 \le m_0 < m_1 < n$, where $S_k = \sum_{i=1}^{k} x_i$. For some $b = m^{1/2}, m_0 = mt_0$ and $m_1 = mt_1$ with $0 < t_0 < t_1 < 1$ and $c > 0$ fixed, he also found out the boundary crossing probability as $n \to \infty$,

$$P_0(S_n \ge b) \sim \frac{1}{4} b^3 \phi(b) \int_{t_0}^{t_1} \frac{1}{(1-t)^2 t^2} \left[\nu \left\{ \frac{c}{(t(1-t))^{1/2}} \right\} \right]^2 dt,$$

where ϕ is a standard normal density function and $\nu(x) = \exp(-0.583x) + o(x^2)$ for small x.

2.2 Lombard statistic

Lombard(1987) proposed the nonparametric test procedures based on quadratic form of rank statistics to test for one or more changepoints in series of independent observations. Let ψ be a square integrable score function on $(0,1)$. Then we can define the standardized rank score such that

$$S(r_i) = \left\{ \psi\left(\frac{r_i}{n+1}\right) - \mu_\psi \right\} / \sigma_\psi,$$

where μ_ψ and σ_ψ are mean and standard deviation of score function ψ, respectively. Denote $r_i = rank(X_i)$. In case of two change-points model, the test statistic for testing H_0 is

$$L_n = 2n \sum_{i=1}^{n-1} \left\{ \sum_{i=1}^{t} s(r_i) \right\}^2 - \left\{ \sum_{j=1}^{n-1} \sum_{i=1}^{j} S(r_i) \right\}^2 .$$

He also found out the following approximation and obtained the asymptotic significance points.

$$n^{-3} L_n \to^D \sum_{n=1}^{\infty} \lambda_n \; Z_n^{\,2},$$

where Z is a standard normal random variable and $\lambda_1 > \lambda_2 > \ldots$ are positive solutions of the equation such that $\{ (2\lambda)^{1/2} \sin (2\lambda)^{-1/2} + \cos (2\lambda)^{-1/2} \} \sin (2\lambda)^{-1/2} = 0.$

2.3 Proposed test statistic

The following statistic can be obtained from Pettitt(1979) statistic. In order to fit this epidemic model, proper modification was made. The proposed test statistic for testing (1) is

$$W_n = n^{-1} \left(\frac{12}{n+1}\right)^{1/2} \max_{1 \le p < q < n} | V_{p,q} - E_0(V_{p,q}) |,$$

where $V_{p,q} = \sum_{i=p+1}^{q} r_i$. To find out the limiting null distribution of W_n , first, we have to derive the moment structure of $V_{p,q}$ as following:

$$E_0(V_{p,q}) = (q-p)(n+1)/2,$$

$$Var_0(V_{p,q}) = (q-p)(n-q+p)(n+1)/12,$$

$$Cov(V_{p,q} , V_{r,s}) = (q-p)(n+1)(n-s+r)/12 ,$$

where $1 \le p < q < n$ and $1 \le q-p \le s-r$. Let B_n be stochastic process written by

$$B_n = \{ B_n(u) , 0 \le u \le 1 \},$$

where

$$B_n(u) = n^{-1} \left(\frac{12}{n+1}\right)^{1/2} \{ V_{[un],[un]+q-p} - E_0(V_{[un],[un]+q-p}) \}$$

with $B_n(0) = B_n(1) = 0$ and $[x] = inf\{k : k \ge x , k \in (0,1,2,\ldots)\}$.

For arbitrary integers $1 \le p < q < r < s < n$, if we denote $u = (q-p)/n$ and $v = (s-r)/n$, then u and v are continuous random variables on $0 < u < v < 1$. It is not difficult to derive the mean and covariance of $B_n(u)$ and $B_n(v)$.

$$E_0(B_n(u)) = 0,$$

$$Cov_0\{ B_n(u), B_n(v) \}$$

$$= 12 \; (n+1)^{-1} \; n^{-2} \; Cov_0(V_{[un],[un]+q-p}, V_{[un],[un]+s-r})$$

$$= \frac{(q-p)}{n} \frac{n-(s-r)}{n}$$

$$= u(1-v).$$

Hence stochastic process B_n and standard Brownian bridge have same moment structure. By the Theorem of Lombard(1983), as $n \to \infty$,

$$W_n \quad \to^{D} \quad \sup_{0 \le u \le 1} \mid B(u) \mid .$$

This is the limiting distribution of the Kolmogorov-Smirnov goodness of fit statistic. Doob(1949) found out the following tail probability of supremum of Brownian Bridge process.

$$P\,[\ \sup_{0 \le u \le 1} \mid B_n(u) \mid \ > x\ ] \doteq 2\ \exp(-2\,x^2\ ). \tag{2}$$

But the results of a simulation study based on 10,000 Monte Carlo trials indicate that the approximations are unsatisfactory for small sample size. In following Table 1, empirical estimates for critical value x are compared with theoretical estimates by using (2). In general, there are two types of statistics to test the change-points. They are sum-type and max-type statistics. It is known that approximations of max-type statistics are less satisfactory than sum-type statistics.

Table 1. Accuracy of approximation

size of test	sample size	empirical estimate					theoretical estimate
		distribution					
		uniform	normal	exponential	Cauchy	double exp.	
0.1	30	1.4206	1.5865	1.6280	1.6280	1.6694	
	50	1.4455	1.5910	1.5958	1.5910	1.6201	1.2238
	100	1.4752	1.6028	1.5356	1.5717	1.5821	
0.05	30	1.5346	1.6902	1.7524	1.7109	1.8043	
	50	1.5425	1.7559	1.6880	1.7365	1.7414	1.3581
	100	1.6045	1.7648	1.6907	1.6958	1.6924	
0.01	30	1.7420	1.9598	1.9598	2.0427	2.032	
	50	1.7608	1.9936	1.9402	1.9402	2.003	1.6276
	100	1.8234	1.9699	1.8975	1.9095	1.8975	

3. Power comparisons of competing tests

A simulation was made to compare the different three test statistics which are parametric likelihood ratio statistic(S_n) and two nonparametric statistics (L_n, W_n) by using the empirical powers. But it is very difficult to find the closed form of power function for the epidemic alternatives. Power function for the epidemic alternatives depends upon significance level of test, null distribution, sample size, two change-points, magnitudes of change and test statistics. The empirical powers were obtained from Monte Carlo simulation where random samples from underlying five null distributions (uniform, normal, exponential, Cauchy and double exponential) were generated by the subroutines of IMSL and 1,000 replications were performed. The significance levels of all tests and sample sizes were fixed on 0.05 and 50, respectively. Because of unsatisfactory approximations of proposed test statistics, we considered the empirical significance level $\alpha=0.05$. Among the descending orders of 1,000 empirical values of the three test statistics under null distributions, we took the 50th value, *i.e.* the 50th largest one, as the critical point. For values of (p,q) in the set

$$\{(5,10),(5,15),\ldots,(10,15),(10,20),\ldots,(40,45)\},$$

$p+50-q$ samples were first generated from $F(x)$, and then $q-p$ additional samples were generated from $F(x-\Delta)$, where $\Delta=\xi_2-\xi_1$ is in epidemic alternative (1). But only powers on the case of $p\in\{10,20\}$ are given here to save space. This values of Δ which are tabulated in Table 2 can be obtained from the equation $P(X_{p+1}>X_p)=0.6$, 0.7, 0.8 and 0.9 where X_p and X_{p+1} have distribution function $F(x)$ and $F(x-\Delta)$, respectively.

Table 2 Estimated Δ values satisfying the equation

distributions	$P(X_{p+1}>X_p)=$			
	0.6	0.7	0.8	0.9
uniform	0.1060	0.2250	0.3680	0.5530
normal	0.3606	0.7425	1.1995	1.8219
exponential	0.2231	0.5108	0.9163	1.6094
Cauchy	0.6498	1.4531	2.7528	6.1553
double exponential	0.4094	0.8731	1.4661	2.3973

The first value of empirical power in Appendix Table A1 is 0.250. This is an empirical power of Sigmund statistic when $n=50, p=10, q=15$, underlying distribution is uniform and magnitude of shift is 0.7. From Appendix Table A1 to Table A2, we can figure out that all empirical powers of tests are increased by the magnitude of change. It means that power functions of tests are monotone increasing. We are also interested in what number of p and q obtain the best powers. The set of pairs of change-points such as

$$\{(p,q): (5,30),(10,35),(15,40),(20,45),(25,45),(30,45),(35,45),(40,45)\}$$

make the best powers among the each null distributions. In above set, $(p,q)=(20,45)$ attains the best power. It means that when $q-p=25$, *i.e.*, half of sample size and center point of series of random samples lies between p and q. Of course, this is not suprising since it is easy to detect the change-points when the two sample sizes are equal.

The likelihood ratio statistic S_n is the best when underlying distribution is uniform or normal. It is natural because test based on S_n is assumed to be normal. But when null distribution is exponential, it is shown that S_n has very poor power. Roughly speaking, proposed test statistic W_n is more powerful than S_n and W_n when distribution is exponential, Cauchy and double exponential. Even though undelying distribution is normal, there is no notable difference between likelihood test and proposed test.

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APPENDIX

Table A1. Empirical powers of size 0.05 tests for epidemic alternatives
($n=50$, $p=10$)

q and statistics		distributions														
		uniform			normal			exponential			Cauchy			double exponential		
		shift			shift			shift			shift			shift		
		0.7	0.8	0.9	0.7	0.8	0.9	0.7	0.8	0.9	0.7	0.8	0.9	0.7	0.8	0.9
15	S	.250	.578	.919	.119	.319	.760	.054	.057	.089	.051	.103	.187	.060	.088	.297
	L	.130	.181	.258	.073	.112	.165	.077	.110	.182	.074	.116	.177	.080	.118	.172
	W	.110	.192	.329	.073	.121	.198	.116	.185	.283	.074	.118	.209	.096	.147	.215
20	S	.470	.865	.999	.250	.626	.977	.053	.062	.179	.103	.276	.594	.075	.181	.764
	L	.208	.404	.691	.141	.299	.536	.141	.302	.575	.145	.294	.566	.159	.324	.607
	W	.281	.667	.957	.210	.480	.860	.277	.626	.952	.215	.493	.837	.248	.534	.897
25	S	.572	.957	1.000	.355	.793	.996	.052	.069	.326	.178	.398	.864	.094	.300	.932
	L	.281	.602	.932	.202	.510	.884	.234	.546	.887	.230	.518	.857	.263	.591	.921
	W	.487	.904	.998	.374	.786	.989	.478	.883	.998	.419	.765	.977	.443	.833	.995
30	S	.654	.977	1.000	.443	.862	.999	.052	.062	.451	.151	.454	.902	.107	.389	.972
	L	.291	.687	.974	.253	.623	.971	.281	.662	.963	.289	.636	.941	.332	.718	.981
	W	.607	.953	1.000	.483	.885	.998	.585	.934	.999	.516	.863	.994	.562	.919	.997
35	S	.654	.969	1.000	.446	.884	.999	.049	.063	.509	.132	.487	.923	.113	.412	.978
	L	.243	.661	.973	.247	.665	.987	.283	.685	.965	.291	.644	.951	.327	.747	.994
	W	.647	.955	1.000	.523	.915	.998	.613	.939	1.000	.540	.882	.997	.590	.931	.998
40	S	.638	.965	.944	.455	.898	1.000	.049	.059	.448	.120	.356	.879	.105	.386	.973
	L	.149	.525	1.000	.206	.614	.962	.236	.598	.936	.257	.591	.934	.307	.692	.984
	W	.626	.950	1.000	.508	.906	.998	.574	.904	.994	.524	.877	.998	.563	.914	.997
45	S	.549	.930	1.000	.396	.842	.996	.047	.054	.314	.108	.327	.766	.090	.323	.941
	L	.106	.357	.825	.219	.556	.911	.197	.498	.846	.240	.527	.879	.290	.622	.936
	W	.498	883	1.000	.417	.833	.996	.478	.819	.977	.433	.797	.984	.461	.837	.996

Table A2. Empirical powers of size 0.05 tests for epidemic alternatives ($n=50, p=20$)

q and statistics		distributions														
		uniform			normal			exponential			Cauchy			double exponential		
		shift			shift			shift			shift			shift		
		0.7	0.8	0.9	0.7	0.8	0.9	0.7	0.8	0.9	0.7	0.8	0.9	0.7	0.8	0.9
25	S	.263	.605	.926	.137	.349	.767	.052	.059	.083	.051	.133	.148	.062	.082	.308
	L	.067	.086	.110	.060	.069	.094	.065	.082	.102	.058	.065	.097	.082	.108	.131
	W	.144	.239	.378	.093	.154	.235	.142	.241	.349	.119	.182	.269	.122	.177	.294
30	S	.482	.870	.993	.262	.667	.977	.052	.061	.175	.086	.343	.345	.077	.176	.770
	L	.085	.161	.347	.093	.186	.409	.100	.205	.392	.110	.224	.426	.145	.279	.534
	W	.332	.687	.963	.234	.519	.890	.332	.617	.949	.258	.513	.842	.281	.553	.898
35	S	.593	.953	1.000	.35.6	.827	.999	.052	.060	.322	.107	.464	.876	.089	.308	.929
	L	.111	.338	.810	.176	.456	.879	.155	.442	.835	.210	.451	.817	.245	.547	.907
	W	.517	.889	.996	.402	.809	.995	.491	.861	.997	.419	.784	.983	.429	.815	.994
40	S	.643	.974	1.000	.452	.914	1.000	.050	.059	.423	.097	.443	.808	.092	.383	.965
	L	.169	.571	.960	.296	.691	.984	.262	.650	.962	.311	.685	.970	.384	.760	.984
	W	.597	.951	1.000	.526	.928	1.000	.591	.911	.998	.531	.886	.999	.557	.911	1.000
45	S	.641	.977	1.000	.480	.916	1.000	.050	.054	.477	.065	.376	.745	.090	.404	.977
	L	.303	.757	.991	.454	.830	.996	.406	.783	.987	.437	.883	.997	.521	.877	.995
	W	.632	.968	1.000	.560	.941	1.000	.605	.930	.999	.578	.921	.999	.596	.937	1.000