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The Counting Process of Which the Intensity Function Depends on States

Jeong Hyun Park[1)]

Abstract

In this paper we are concered with the counting processes with intensity function $g_n(t)$, where $g_n(t)$ not only depends on t but n. It is shown that under certain conditions the number of events in $[0, t]$ follows a generalizes Poisson distribution. A counting process is also provided such that $g_i(t) \neq g_j(t)$ for $i \neq j$ and the number of events in $[0, t]$ has a transformed geometric distribution.

1. Introduction

The Poisson distribution has been generalized in many ways. Rao and Rubin (1964), Chon (1960) and Singh (1966) introduced a generalized Poisson distribution. Consul and Jain (1970) presented a new generalized Poisson distribution with two parameter (θ, λ). Recently, Consul (1988) investigated some models leading to the generalized Poisson distribution and Consul (1989) introduced a generalized Poisson process for which the number of events in interval [0, t] is generalized Poisson distributed with parameters $(\theta t, \lambda t)$.

In Poisson processes, the intensity function is $g_n(t) = \lambda$ for all $n = 1, 2, \cdots$, and the intensity function of nonhomogeneous Poisson process is $g_n(t) = \lambda(t)$ for all $n = 1, 2, \cdots$. Therefore, the Poisson process and nonhomogeneous Poisson process have intensity function which does not depend on state n. Consul (1989) proposed a generalized Poisson process of which that intensity function depends on state n.

In this paper, we provide some conditions under which the number of events in $[0, t]$ follows a generalized Poisson distribution, and show that the definition of a generalized Poisson process defined by Consul(1989) is not well-defined. In section 2, we propose a counting process such that the intensity function is $g_i(t) \neq g_j(t)$ for $i \neq j$.

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1) Associate Professor, Department of Computer Science and Statistics, Kwandong University, Kangnung, 210-701

Let $\{N(t) \mid t \geq 0\}$ be a counting process having jump magnitute 1. Then counting process $\{N(t) \mid t \geq 0\}$ is satisfies

$$P\{N(t+h)-N(t) \geq 2\} = o(h). \tag{1.1}$$

Suppose that $P\{N(t+h)-N(t) = 1 \mid N(t)=n\} = g_n(t, h)$ and $g_n(t, h)$ is a polynomial function with respect to h in which the constant term is zero;

$$g_n(t, h) = g_n(t)\, h + g_n(t)\, h^2 + \cdots = g_n(t)\, h + o(h).$$

Then

$$P\{N(t+h)-N(t) = 1 \mid N(t)=n\} = g_n(t)\, h+o(h) \tag{1.2}$$

and

$$P\{N(t+h)-N(t) = 0 \mid N(t)=n\} = 1 - g_n(t)\, h+o(h)...$$

Now $g_n(t)$ is called the *intensity function* of counting process $\{N(t) \mid t \geq 0\}$.

By Equations (1.1) and (1.2),

$$\begin{aligned} P\{N(t+h)=n\} &= P\{N(t+h)-N(t)=0 \mid N(t)=n\}P\{N(t)=n\} \\ &\quad +P\{N(t+h)-N(t)=1 \mid N(t)=n-1\}P\{N(t)=n-1\} \\ &\quad + \sum_{i=2}^{\infty} P\{N(t+h)-N(t)=i \mid N(t)=n-i\}P\{N(t)=n-i\} \\ &= \{1-g_n(t)\, h\}\, P\{N(t)=n\}+g_{n-1}(t)\, hP\{N(t)=n-1\}+o(h). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{P\{N(t+h)=n\}-P\{N(t)=n\}}{h} &= g_n(t)P\{N(t)=n\} \\ &\quad + g_{n-1}(t)P\{N(t)=n-1\}+\frac{o(h)}{h}. \end{aligned}$$

Letting $h \to 0$, we obtain the differential equation

$$\frac{dP\{N(t)=n\}}{dt} = g_n(t)\; P\{N(t)=n\} + g_{n-1}(t)\; P\{N(t)=n-1\}.$$

The solution of the above equation is

$$P\{N(t)=n\}=e^{-\int g_n(t)\,dt}\int g_{n-1}(t)P\{N(t)=n-1\}e^{\int g_n(t)\,dt}dt+k_n e^{-\int g_n(t)\,dt}. \tag{1.3}$$

If the countig process is a Poisson or nonhomogeneous Poisson, we know that $k_0 = 1$ and $k_n = 0\ (n \geq 1)$. The constants $\{k_0, k_1, k_2, \cdots\}$ is called the *integral constants* of the corresponding counting process. From (1.3) we can see that the distribution of $N(t)$, the number of events in interval [0, t], depends on $g_n(t)$ and $g_{n-1}(t)$. It is well-known that $g_n(t) = \lambda$ for all n is a necessary condition for the counting process to be a Poisson process with rate λ. Similarly, $g_n(t) = \lambda(t)$ for all n is necessary for the counting process to be a nonhomogeneous Poisson process with intensity function $\lambda(t)$. As can be seen, $g_n(t)$ does not depend on n for the cases of Poisson and nonhomogeneous Poisson processes. In general, however, $g_n(t)$ depends on n in generalized Poisson processes.

2. P-process

Let $\int_* f(t)dt = \int f(t)dt - C$, where C is a integral constant of $f(t)$.

The function $f(t)$ is said to be a *t-zero function* if $\left[\int_* f(t)dt\right]_{t=0} = 0$.

[Definition 1] The counting process $\{N(t) \mid t \geq 0\}$ is said to be a *polynomial process (P-process) with intensity function* $g_n(t)$ if

(i) $N(0) = 0$,

(ii) $P\{N(t+h) - N(t) = 1 \mid N(t) = n\} = g_n(t)h + o(h)$

where $-\infty < \left[\int g_n(t)\,dt\right]_{t=0} < \infty$,

(iii) $P\{N(t+h) - N(t) \geq 2 \mid N(t) = n\} = o(h)$ for each $n = 0, 1, 2, \cdots$.

If $g_n(t) = \lambda$ for each $n = 0, 1, 2, \cdots$, then the P-process is a Poisson process with rate λ, and if $g_n(t) = \lambda(t)$ for each $n = 0, 1, 2, \cdots$, then the P-process is a nonhomogeneous Poisson process with intensity function $\lambda(t)$.

Let $P_n(t) = P\{N(t) = n\}$. Then, from the definition and (1.3), we obtain that

$$P_0(t) = k_0 \exp\left(-\int_* g_0(t)\,dt\right),$$

and for $n \geq 1$,

$$P_n(t) = \exp\left(-\int_* g_n(t)\,dt\right)\left[\int_* g_{n-1}(t)P_{n-1}(t)\exp\left(\int g_n(t)\,dt\right)dt\right] + k_n \exp\left(-\int_* g_n(t)\,dt\right),$$

where $k_0, k_1, \cdots$ are constants.

[Proposition 1] If $\{N(t) \mid t \geq 0\}$ is a P-process with intensity function $g_n(t)$, then

(1) $g_0(t) = -\dfrac{P_0'(t)}{P_0(t)}$.

(2) $g_n(t) = \dfrac{g_{n-1}(t)P_{n-1}(t)}{P_n(t)} - \dfrac{P_n'(t)}{P_n(t)}$, $n \geq 1$.

By the boundary condition $P_0(0) = 1$ and $P_n(0) = 0$ $(n \geq 1)$, we obtain the following :

[Theorem 2] Let $\{N(t) \mid t \geq 0\}$ be a P-process with intensity function $g_n(t)$. Then,

(1) $g_0(t)$ is a t-zero function if and only if $k_0 = 1$.

(2) $g_{n-1}(t)P_{n-1}(t)\exp\left(\int g_n(t)\,dt\right)$ $(n \geq 1)$ is a t-zero function if and only if $k_n = 0$.

(Proof) (1) By the boundary condition $P_0(0) = 1$,

$$
\begin{aligned}
g_0(t) \text{ be a t-zero function } &\Rightarrow \left[-\int_* g_0(t)\,dt\right]_{t=0} = 0 \\
&\Rightarrow P_0(0) = k_0 \exp(0) = 1 \\
&\Rightarrow k_0 = 1 \\
&\Rightarrow P_0(0) = \left[\exp\left(\int_* g_0(t)\,dt\right)\right]_{t=0} = 1 \\
&\Rightarrow g_0(t) \text{ be a t-zero function.}
\end{aligned}
$$

(2) By the boundary condition $P_n(0) = 0$ $(n \geq 1)$,

$g_{n-1}(t)P_{n-1}(t)\exp\left(\int g_n(t)\,dt\right)$ $(n \geq 1)$ is a t-zero function

$$
\begin{aligned}
&\Rightarrow k_n \exp\left(-\int_* g_n(t)\,dt\right) = P_n(0) = 0 \\
&\Rightarrow k_n = 0 \\
&\Rightarrow P_n(0) = \left[\exp\left(-\int_* g_n(t)\,dt\right)\int_* g_{n-1}(t)P_{n-1}(t)\exp\left(\int_* g_n(t)\,dt\right)dt\right]_{t=0} = 0 \\
&\Rightarrow \left[\int_* g_{n-1}(t)P_{n-1}(t)\exp\left(\int_* g_n(t)\,dt\right)dt\right]_{t=0} = 0 \\
&\Rightarrow g_{n-1}(t)P_{n-1}(t)\exp\left(\int g_n(t)\,dt\right) \ (n \geq 1) \text{ is a t-zero function.} \quad \square
\end{aligned}
$$

[Definition 2] The P-process $\{N(t) \mid t \geq 0\}$ is called a *strongly P-process* if

$$k_0 = 1 \text{ and } k_n = 0 (n \geq 1).$$

[Proposition 3] If $\{N(t) \mid t \geq 0\}$ is a strongly P-process with intensity function $g_n(t)$, then

$$g_0(0) = P'_1(0) \text{ and } P'_n(0) = 0 \ (n \geq 2).$$

(Proof) For $n \geq 1$,

$$k_n = \frac{g_{n-1}(0)P_{n-1}(0) - P'_n(0)}{g_n(0)} \left[\exp\left(\int_* g_n(t)\,dt\right)\right]_{t=0} - \left[\int_* g_{n-1}(t)\,P_{n-1}(t)\exp\left(\int_* g_n(t)\,dt\right)dt\right]_{t=0}.$$

Since $\{N(t) \mid t \geq 0\}$ is a strongly P-process, $k_n = 0$. By [Theorem 2],

$$\left[\int_* g_{n-1}(t)\,P_{n-1}(t)\exp\left(\int_* g_n(t)\,dt\right)dt\right]_{t=0} = 0.$$

Hence,

$$\frac{g_{n-1}(0)P_{n-1}(0) - P'_n(0)}{g_n(0)} = 0 ,$$

which implies that

$$g_{n-1}(0)P_{n-1}(0) - P'_n(0) = 0 .$$

Taking $n = 1$. By the boundary condition $P_0(0) = 1$,

$$g_0(0) = P'_1(0).$$

If $n \geq 2$, $P_n(0) = 0$ and thus $P'_n(0) = 0$. □

The Poisson process is a strongly P-process because $g_0(t) = \lambda$ is a t-zero function, and

$$g_{n-1}(t)P_{n-1}(t)\exp\left(\int_* g_n(t)\,dt\right) = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

is a t-zero function. For the case of nonhomogeneous Poisson processes, if intensity function $\lambda(t)$ is a t-zero function, then the nonhomogeneous Poisson process is a strongly P-process.

The following example shows that there exists a strongly P-process such that $g_i(t) \neq g_j(t)$ for some i, j.

[Example 1] Let the counting process $\{N(t) \mid t \geq 0\}$ satisfy

(1) $N(0) = 0$,

(2) $P\{N(t+h) - N(t) = 1 \mid N(t) = 0\} = \left(\theta - \frac{\theta\lambda}{1+\theta\lambda t}\right)h + o(h)$,

(3) $P\{N(t+h) - N(t) = 1 \mid N(t) = 1\} = \left(\frac{\theta}{1-\lambda}\right)h + o(h)$ where $0 \leq \lambda < 1$,

(4) $P\{N(t+h) - N(t) = 1 \mid N(t) = n\} = \lambda h + o(h)$, $n = 2, 3, 4, \cdots$,

(5) $P\{N(t+h) - N(t) \geq 2 \mid N(t) = n\} = o(h)$ for each $n = 0, 1, 2, \cdots$.

Since $g_0(t) = \theta - \frac{\theta\lambda}{1+\theta\lambda t}$ is a t-zero function, $k_0 = 1$ by Theorem 2. and

$$P_0(t) = e^{-\theta t}(1 + \theta\lambda t) .$$

Since

$$g_0(t) P_0(t) \exp\left(\int_* g_1(t)\,dt\right) = (\theta - \theta t + \theta^2\lambda t)\exp\left(\frac{\theta\lambda t}{1-\lambda}\right)$$

is a t-zero function, by Theorem 2, $k_1 = 0$. Thus we obtain

$$P_0(t) = \theta t e^{-\theta t}(1 - \lambda).$$

For $n \geq 2$,

$$g_{n-1}(t) P_{n-1}(t) \exp\left(\int_* g_n(t)\,dt\right) = \frac{\theta^n}{(n-1)!}\, t^{n-1}$$

is a t-zero function. Also ,by Theorem 2, $k_n = 0$ and we obtain

$$P_n(t) = \frac{(\theta t)^n}{n!}\, e^{-\theta t}.$$

Thus counting process $\{N(t) \mid t \geq 0\}$ is a strongly P-process and the distribution of number of events in the interval [0, t] is

$$P\{N(t) = n\} = \begin{cases} e^{-\theta t}(1 + \theta\lambda t), & n = 0 \\ \theta t e^{-\theta t}(1 - \lambda), & n = 1 \\ \frac{(\theta t)^n}{n!} e^{-\theta t}, & n = 2, 3, 4, \cdots. \end{cases}$$

The above distribution is a generalized Poisson distribution with parameters $(\theta t, \lambda)$ defined by Rao and Rubin (1964). For convenience, this strongly P-process is called to be a *(0,1)-generalized Poisson process.* □

The following example is a P-process but not strongly P-process.

[Example 2] Let counting process $\{N(t) \mid t \geq 0\}$ satisfy

(1) $N(0) = 0$,

(2) $P\{N(t+h) - N(t) = 1 \mid N(t) = n\} = \theta h + o(h),\ n = 0, 3, 4, 5, \cdots$,

(3) $P\{N(t+h) - N(t) = 1 \mid N(t) = 1\} = \left(\theta - \dfrac{2\theta\lambda}{2 + \theta\lambda t}\right)h + o(h)$,

(4) $P\{N(t+h) - N(t) = 1 \mid N(t) = 2\} = \left(\dfrac{\theta}{1-\lambda}\right)h + o(h)$ where $0 \leq \lambda < 1$,

(5) $P\{N(t+h) - N(t) \geq 2 \mid N(t) = n\} = o(h)$ for each $n = 0, 1, 2, \cdots$.

Since $g_n(t)$ is a t-zero function except $n = 1$ and

$$\left[\int_{\bullet} g_1(t)\,dt\right]_{t=0} = \left[\int_{\bullet}\left(\theta - \frac{2\theta\lambda}{2 + \theta\lambda t}\right)dt\right]_{t=0} = -2\ln 2 < \infty,$$

we know that counting process $\{N(t) \mid t \geq 0\}$ is a P-process.
By Theorem 2, $g_0(t) = \theta$ is a t-zero function imply $k_0 = 1$. However,

$$\begin{aligned} g_0(t)P_0(t)\exp\left(\int g_1(t)\,dt\right) &= \theta\exp(-\theta t)\exp\left(\int\left(\theta - \frac{2\theta\lambda}{2 + \theta\lambda t}\right)\right) \\ &= -\frac{\theta}{(2 + \theta\lambda t)^2} \end{aligned}$$

is not a t-zero function. Thus $\{N(t) \mid t \geq 0\}$ is not a strongly P-process. In this case we get the integral constants $k_1 = 1/2\lambda$ and $0 = k_2 = k_3 = \cdots$.
The distribution of number of events in the interval [0, t] is the following

$$P\{N(t) = n\} = \begin{cases} e^{-\theta t}, & n = 0 \\ \theta t e^{-\theta t}\left(1 + \dfrac{\theta\lambda t}{2}\right), & n = 1 \\ \dfrac{(\theta t)^2}{2!} e^{-\theta t}(1 - \lambda), & n = 2 \\ \dfrac{(\theta t)^n}{n!} e^{-\theta t}, & n = 3, 4, \cdots. \end{cases}$$

For convenience, we call the above distribution by *(1,2)-generalized Poisson distribution* with parameters $(\theta t, \lambda)$ and the associated P-process *(1,2)-generalized Poisson process* with parameters $(\theta t, \lambda)$. □

Consul and Jain (1973) presented a new generalized Poisson distribution with two parameters θ and λ. The definition of generalized Poisson distribution is as follows :

[Definition] (Consul and Jain) The discrete random variable X is said to be a generalized Poisson distribution with parameters (θ, λ) if a probability mass function of X is

$$P_X(\theta,\lambda) = \begin{cases} \dfrac{\theta(\theta+x\lambda)^{x-1}e^{-\theta-x\lambda}}{x!}, & \text{if } x = 0, 1, 2, \cdots \\ 0, \quad \text{for } x > m, & \textit{when } \lambda < 0 \text{ and } \textit{otherwize} \end{cases}$$

where $\theta > 0$, $\max(-1, -\theta/m) < \lambda \leq 1$ and $m(\geq 4)$ is the largest positive for which $\theta + m\lambda > 0$ when λ is negative.

Consul (1989) introduced a new generalized Poisson process where the number of changes $N(t)$ in an interval $[0, t]$ has a generalized Poisson distribution with parameters $(\theta t, \lambda t)$. He made a suggestion to adopt four axioms of new generalized Poisson process. The four axioms are the following :

(Axiom 1) N(0) = 0.

(Axiom 2) For any $t > 0$, $0 < P\{N(t) > 0\} < 1$.

(Axiom 3) Direct transitions from state i are possible only to state $i+1$ and not to the higher states; that is, for any $t > 0$, in sufficiently small intervals of duration h, at most one event can occur.

(Axiom 4) $P\{N(t+h) - N(t) = 0 \mid N(t) = n\} = 1 - (\theta + n\lambda)h + o(h)$

$$P\{N(t+h) - N(t) = 1 \mid N(t) = n-1\} = \frac{(\theta + (n-1)\lambda)he^{-\lambda t}}{(1 - \lambda(\theta + n\lambda)^{-1})^{n-1}} + o(h).$$

From Axioms, we know that a new generalized Poisson process is a P-process. Consul insisted that the integral constants of new generalized Poisson process are $k_0 = 1$ and $k_n = 0\ (n \geq 1)$, and the distribution of number of events in interval [0, t] is a new generalized Poisson with parameters $(\theta t, \lambda t)$. Therefore if Consul's results are correct, the new generalized Poisson process is a strongly P-process. But, under the above Axioms, we cannot obtain that the distribution of number of events in interval [0, t] is a new generalized Poisson with parameters $(\theta t, \lambda t)$. Moreover, the definition of new generalized Poisson process is not well-defined.

Suppose the above counting process is a strongly P-process and the distribution of number of events in interval [0, t] is a new generalized Poisson with parameters $(\theta t, \lambda t)$.

Then, by Proposition 1,

$$g_0(t) = -\frac{P_0'(t)}{P_0(t)}$$

and

$$g_n(t) = \frac{g_{n-1}(t)P_{n-1}(t)}{P_n(t)} - \frac{P_n'(t)}{P_n(t)}, \quad n \geq 1.$$

But, since $g_0(t) = \theta$, $P_0(t) = e^{-\theta t}$ and $P_1(t) = \theta t e^{-\theta t - \lambda t}$,

$$g_1(t) = \frac{1}{t}(e^{\lambda t} - 1) + (\theta + \lambda).\ .$$

Thus $g_1(t) = \frac{1}{t}(e^{\lambda t} - 1) + (\theta + \lambda)$ is not equal to the result of Axiom 4.

Hence, the counting process of which the distribution of number of events in interval [0, t] is a new generalized Poisson with parameters $(\theta t, \lambda t)$ is not a P-process. It is because $g_1(t)$ does not satisfy

$$-\infty < \left[\int g_1(t)\,dt\right]_{t=0} < \infty.$$

Now, we show that Axiom 4 is contradicted to Axiom 3.

From Axiom 3,

$$P\{N(t+h) - N(t) \geq 2 \mid N(t) = x\} = o(h) \quad \text{for each } x = 0, 1, 2, \cdots .$$

Note that

$$\begin{aligned} 1 &= \sum_{i=0}^{\infty} P\{N(t+h) - N(t) = i \mid N(t) = x\} \\ &= P\{N(t+h) - N(t) = 0 \mid N(t) = x\} + P\{N(t+h) - N(t) = 1 \mid N(t) = x\} \\ &\qquad + P\{N(t+h) - N(t) \geq 2 \mid N(t) = x\}. \end{aligned}$$

Then

$$\begin{aligned} o(h) &= P\{N(t+h) - N(t) \geq 2 \mid N(t) = x\} \\ &= 1 - P\{N(t+h) - N(t) = 0 \mid N(t) = x\} - P\{N(t+h) - N(t) = 1 \mid N(t) = x\}. \end{aligned}$$

Taking $x = 0$, by Axiom 4,

$$\begin{aligned} &P\{N(t+h) - N(t) \geq 2 \mid N(t) = 0\} \\ &= 1 - P\{N(t+h) - N(t) = 0 \mid N(t) = 0\} - P\{N(t+h) - N(t) = 1 \mid N(t) = 0\} \\ &= \theta h - \theta h e^{-\lambda t} + o(h) \\ &\neq o(h). \end{aligned}$$

This is a contradiction to Axiom 3. Therefore, the definition of a generalized Poisson process is not well-defined.

3. Transformed Geometric Poisson process

Let X be a geometric random variable. Then random variable $Y = X - 1$ is called to be *transformed geometric*. In this section, we introduce a P-process for which the distribution of number of events in interval [0, t] is transformed geometric and $g_i(t) \neq g_j(t)$ for each $i \neq j\,(i, j = 0, 1, 2, \cdots)$.

[Definition 3] The P-process $\{N(t) \mid t \geq 0\}$ is said to be a *transformed geometric Poisson process* with intensity function $f(t)$ if

(i) $f(0) = 0$

(ii) $0 \leq f(t) < 1$ for each $t \geq 0$

(iii) $g_n(t) = (n+1)\dfrac{df(t)/dt}{1 - f(t)}$.

[Theorem 3] Let $\{N(t) \mid t \geq 0\}$ be a transformed geometric Poisson process with intensity function $f(t)$. Then the number of events in interval [0, t] is a transformed geometric distribution with parameter $(1 - f(t))$ and the integral constants of $\{N(t) \mid t \geq 0\}$ are $k_n = (-1)^n$.

(Proof) Since $P_0(t) = k_0 \exp\left(-\int_* g_0(t)\,dt\right) = k_0(1 - f(t))$ and

$$g_0(t) = \frac{df(t)/dt}{1 - f(t)}$$

is a t-zero function, $k_0 = 1$ by [Theorem 2].

Hence,

$$P_0(t) = (1 - f(t)).$$

$$\begin{aligned} P_1(t) &= \exp\left(-\int_* g_1(t)\,dt\right)\left[\int_* g_0(t)P_0(t)\exp\left(\int_* g_1(t)\,dt\right)\right] + k_1 \exp\left(-\int_* g_1(t)\,dt\right) \\ &= (1 - f(t))^2 \int_* \frac{f'(t)}{(1 - f(t))^2}\,dt + k_1(1 - f(t))^2 \\ &= (1 - f(t)) + k_1(1 - f(t))^2. \end{aligned}$$

The boundary condition $P_1(0)=0$ implies that $k_1=-1$. Then,

$$P_1(t) = f(t)(1-f(t)).$$

To show that $P_n(t)=f(t)^n(1-f(t))$, we use the mathematical induction.

Suppose $P_{n-1}(t) = f(t)^{n-1}(1-f(t))$. Then

$$P_n(t) = \exp\left(-\int_* g_n(t)\,dt\right)\left[\int_* g_{n-1}(t)P_{n-1}(t)\exp\left(\int_* g_n(t)\,dt\right)dt\right] + k_n\exp\left(-\int_* g_n(t)\,dt\right)$$

$$= (1-f(t))^{n+1}\int_* \frac{nf'(t)f(t)^{n-1}}{(1-f(t))^{n+1}}\,dt + k_n(1-f(t))^{n+1}$$

$$= \sum_{i=1}^{n}(-1)^{i+1}f(t)^{n-i}(1-f(t))^i + k_n(1-f(t))^{n+1}.$$

The boundary condition $P_n(0)=0$ implies that $k_n=(-1)^n$. Hence, we obtain

$$P_n(t) = f(t)^n(1-f(t)). \qquad \square$$

[Example 3] Let $f(t)=1-e^{-\theta t}$. Then $f(0)=0$ and $0\le f(t)<1$.

Taking $g_n(t) = (n+1)\dfrac{df(t)/dt}{1-f(t)}$,

$$g_n(t) = (n+1)\theta$$

which implies

$$P_n(t) = (1-e^{-\theta t})^n e^{-\theta t}.$$

Therefore, the number of events in interval [0, t] is a transformed geometric distribution with parameter $f(t)=1-e^{-\theta t}$. $\square$

The following diagram represents that the relation of the generalized Poisson process with $g_n(t)$ and the integral constants.

Strongly P-process $\Leftarrow$ Poisson $\Longleftrightarrow$ $g_n(t)=\lambda$
($k_0=1,\ 0=k_1=k_2=\cdots$)

Strongly P-process $\Leftarrow$ nonhomogeneous Poisson $\Longleftrightarrow$ $g_n(t)=\lambda(t)$
($k_0=1,\ 0=k_1=k_2=\cdots$)

Strongly P-process $\Leftarrow$ (0,1)-generalized Poisson $\Longleftrightarrow$ $g_n(t) = \begin{cases} \theta - \dfrac{\theta\lambda}{1+\theta\lambda t}, & n=0 \\ \dfrac{\theta}{1-\lambda}, & n=1 \\ \theta, & n \geq 2 \end{cases}$

$(k_0 = 1,\ 0 = k_1 = k_2 = \cdots)$

P-process $\Leftarrow$ (1, 2)-generalized Poisson $\Longleftrightarrow$ $g_n(t) = \begin{cases} \theta - \dfrac{2\theta\lambda}{2+\theta\lambda t}, & n=1 \\ \dfrac{\theta}{1-\lambda}, & n=2 \\ \theta, & n=0, 3, 4, \cdots \end{cases}$

$(k_0 = 1,\ k_1 = 1/2\lambda,\ 0 = k_2 = \cdots)$

P-process $\Leftarrow$ transformed geometric Poisson $\Longleftrightarrow$ $g_n(t) = (n+1)\dfrac{df(t)/dt}{1-f(t)}$

$(k_n = (-1)^n)$.

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