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A Special Case of a Two-Sex Model in the Growth of Population

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Abstract

We consider two models for the growth of population with overlaping generations. First, the model we will describe is basically the model given by Leslie(1945). This is only a one-sex model of population age structure and growth. Next, we introduce a model in which couples must be formed before reproduction occurs. If the maximum number of couples is formed, and if the couples are only formed from females of age x-a and males of age x at time t, $a>0$. Then, we will solve the renewal equations for the reproductive value.

1. Introduction

Population growth models start with the simple exponential model and proceed to more elaborate forms that propose to represent the trajectory of populations. The exponential model applies to a population in which all rates are fixed. Recognizing overall growth at fraction $r=b-d$, where b is the crude birth rate, d is the crude death rate, both applying over the short period dt, we have for the population $P(t)$ at time t the equation $dP(t)=rP(t)dt$, whose integral is $P(t)=P_0e^{rt}$. Conversely, if we know the population number, the rate of growth is expressible as $r=\ln(P(t)/P_0)/t$, where ln stands for the natural logarithm. A model in which growth is an intermediate stage between two stationary conditions was developed by Verhulst(1838), and later rediscovered by Pearl(1939) and others. The age-dependent population model which was introduced by Lotka(1907) and Cole(1954), McKendrick(1926), Feller(1941), Foerster(1959) expressed in other equivalent ways. Sex is more difficult to model than age. Pollard(1973) and Schoen(1978) tried to show the evolution of a two-sex population continue, inevitably based on simplyfying assumptions whose realism is not easily tested. A frequent question concerns the growth of a population in which not only age and sex but labor force status, geographical sub areas, and other characteristic are

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recognized. A method for this is due to Rogers(1975) and Schoen(1975). Also, Waugh(1955) extended the mathematics to the age-dependent case, and Pollard(1966) created a branching process model that incorporates variances and covariances in an elegant extension of the Leslie(1945) matrix. In this paper, we derive a formular of the reproductive value of an ancestral female in age group x in Section 2 and derive formulars of the reproductive values for an ancestral couple in female age group x-a and male age group x with respect to the growth of females and males in Section 3.

2. Classical Model

Classical stable population theory is only a one sex model of population age structure and growth. We will suppose that the population consists of only one type of individual, say female, and that only female offspring are produced. The model we will now describe is basically the model given by Leslie(1945). We will say that a female belongs to age group x at time t, if she is between x and x+1 units of age at that time.

Let $F(t,x)$: number of females in age group x at time t.

$P(x)$: P(any female survives from birth to age group x).

$b_1(x)$: E(number of female offspring a female parent in age group x at time t contribute to age group 0 at time t+1).

2.1 Renewal equation

Let n be the maximum age of survival for any individual. Thus, if x>n then P(x) = 0. Also, let s be the maximum age in which reproduction occurs: that is, $b_1(x)=0$ if $x > s$. Thus, the renewal equation is :

$$F(t,0) = \sum_{x=0}^{s} b_1(x)F(t-1,x), \qquad t=s+2,s+3,\cdots, \tag{2.1}$$

where $F_0(t)$ denotes the female offspring of the ancestral individuals alive at time t for t = 0, 1, 2, · · · , s.

Also, $F(t,x)=\begin{cases} F(t-x,0)P(x), & x\le t \\ F(0,x-t)P(x)/P(x-t), & x>t \end{cases}$ •

Thus, if we use this equation in (2.1), we obtain

$$F(t,0)= \sum_{x=0}^{s} b_1(x)P(x)F(t-x-1,0), \qquad t=s+2,s+3,\ldots \quad . \tag{2.2}$$

If the parameters $b_1(x)$ and P(x) are uniform in time then

$$F_0(t) = \sum_{y=0}^{s-t} b_1(y+t)F(0,y)P(y+t)/P(y), \qquad t=0,1,\ldots,s$$
$$= 0, \qquad \text{otherwise} \quad .$$

The formulation of discrete stable population theory was made given by Leslie(1945). Let $P_x = P$(any female in age group x at time t survives to age group x+1 at time t+1). Thus

$$F(t,0) = \sum_{y=0}^{s} b_1(x)F(t-1,y)$$
$$F(t,1) = F(t-1,0)P(1) = F(t-1,0)P_0$$
$$F(t,2) = F(t-2,0)P(2) = F(t-2,0)P_0P_1 = F(t-1,1)P_1$$
$$\vdots$$
$$F(t,n) = F(t-1,n-1)P_{n-1} \quad \text{for } t=1,2,\cdots \quad .$$

These equtions are expressed in matrix form. Say,

$F(t) = LF(t-1)$, $t=1,2,\cdots$, where, $F(t) = \{F(t,0), F(t,1), \cdots F(t,s)\}'$ and L is a square matrix with elements

$$L = \{l_{xy}\} = \begin{cases} b_1(y), & \text{if } x=0 \\ P_y, & \text{if } x=y+1 \\ 0, & \text{otherwise} \end{cases} , \quad \text{for } x,\ y = 0,\ 1,\ \ldots,\ s\ .$$

Thus, we have that

$$F(t) = L^t F(0), \quad t=0,1,2 \quad \cdots \quad . \tag{2.3}$$

Next, we derive a reproductive value of an ancestral female in age group x by using the above equations.

2.2 Reproductive value

If we take the Laplace transform of Equation (2.2) we have that

$$\sum_{t=1}^{\infty} e^{-rt}F(t,0) = \sum_{t=1}^{\infty} e^{-rt}F_0(t-1) + \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} e^{-r(x+1)}b_1(x)P(x)\sum_{y=1}^{\infty} e^{-ry}F(y,0),$$

where y=t-x-1. Thus,

$$\sum_{y=1}^{\infty} e^{-ry}F(y,0) = \frac{\sum_{y=1}^{\infty} e^{-ry}F_0(y-1)}{1-\sum_{x=0}^{\infty} e^{-r(x+1)}b_1(x)P(x)} \quad . \tag{2.4}$$

If $e^{-ry} = \eta^y$ then Equation (2.4) is

$$\sum_{y=1}^{\infty} \eta^y F(y,0) = \frac{\sum_{y=1}^{\infty} F_0(y-1)}{1-\sum_{x=0}^{\infty} \eta^{(x+1)}b_1(x)P(x)} \quad . \tag{2.5}$$

The matrix L of Equation (2.3) is a nonnegative matrix(Leslie(1945)). If fertility is not a periodic function of age, then this matrix is also irreducible. The results of Definitions and Theorems by Sykes(1969) are applicable. Thus, we let $\eta^{-1}=\mu_0$, then obtain $F(y,0)\approx F\mu_0^t$, as $y\rightarrow\infty$, where μ_0 is the simple dominant positive root of

$$\phi(\mu)=1-\sum_{x=0}^{s}\mu^{-(x+1)}b_1(x)P(x), \quad \text{and}$$

$$F=\frac{\sum_{x=1}^{s+1}\mu_0^{-x}F_0(x-1)}{1-1+\sum_{x=0}^{s}(x+1)\mu_0^{-(x+1)}b_1(x)P(x)}=\frac{\sum_{x=1}^{s+1}\mu_0^{-x}F_0(x-1)}{\sum_{x=0}^{s}(x+1)\mu_0^{-(x+1)}b_1(x)P(x)}$$

Since $F(t,x)=F(t-x,0)P(x), \quad t\geq x$,

$F(t,x)=F(t-x,0)P(x) \approx F\mu_0^{t-x}P(x), \quad t\rightarrow\infty$, for x= 0, 1, 2, ..., n .

This means that the age distribution ultimately stabilizes. Now,

$$F=\nu\sum_{y=0}^{s}F(0,y)\sum_{x=1}^{s-y+1}\mu_0^{-x}b_1(y+x-1)P(y+x-1)/P(y),$$

where $\frac{1}{\nu}=\sum_{x=0}^{s}(x+1)\mu_0^{-(x+1)}b_1(x)P(x)$.

We define

$\nu(x)$ = measure of reproductive value of an ancestral female in age group x

= coefficient of F(0,x) in F .

Therefore,

$$\nu(z)=\text{coefficient of } F(0,x) = \frac{\nu\mu_0^z}{P(z)}\sum_{y=z}^{s}\mu_0^{-(y+1)}b_1(y)P(y),$$

where y=z+x-1.

Result:

The reproductive value of an ancestral female in age group x is

$$\nu(x)=\frac{\nu\mu_0^x}{P(x)}\sum_{y=x}^{s}\mu_0^{-(y+1)}b_1(y)P(y), \quad x=0,1,\dots,n$$
$$=0, \quad x=s+1,s+2,\dots,n,$$

where we have defined $\nu(x)$ = 0 if x>s, since any female of age greater than s does not reproductive. We also consider the total reproductive value of females alive at time t. That is:

$$V(t)=\sum_{x=0}^{s}F(0,x)\nu(x), \qquad t\geq 0 .$$

3. A Special case of a two-sex model

We now state the assumptions for a two-sex model that we are going to study. The assumptions are :

(1) males of age x mate only with females of age x-a,

(2) at birth, the ratio of females to males is a known constant,

(3) monogamous mating .

Individuals in the population are classified by age, type and time. We will say that an individual belongs to age group x at time t, if he or she is between x and x+1 units of age at that time.

Let $F(t,x-a)$ = number of females in age group x at time t.

$M(t,x)$ = number of males in age group x at time t.

$C(t, x-a,x)$ = number of couples formed at time t with the female of age x-a and the male age x.

$P(x-a)$ = P(any female survives from birth to age group x-a).

$Q(x)$ = P(any male survives from birth to age group x).

$b_1(x-a,x)$ = E(number of female offspring a couple in male age group x and female age group x-a at time t contributes to age group 0 at time t+1).

$b_2(x-a,x)$ = E(number of male offspring a couple in male age group x and female age group x-a at time t contributes to age group 0 at time t+1).

Assume that $F(t,x-a)$ and $M(t,x)$ are unknown among the expressions that have just been defined, but the other expressions are known and $b_1(x-a,x)$ and $b_2(x-a,x)$ are nonnegative numbers. Let n be the maximum age to which any female or male can survive. Say, $P(x)=0$ if $x>n$, $Q(x)=0$ if $x>n$. and let s be the maximum age group in which reproduction can occur. Thus, if the male age is greater than s, then $b_1(x-a,x)$ and $b_2(x-a,x)$ are zero for x= a, a+1, ...

3.1 Renewal Equations

The number in female age group 0 at time t is $F(t,0)$ and the number in male age group at that time is $M(t,a)$. Thus, the renewal equations are :

$$\begin{aligned} F(t,0) &= F^*(t-1)+\sum_{x=a}^{t-2} b_1(x-a,x)C(t-1,x-a,x), \quad t=1,2,\dots,s+1 \\ &= \sum_{x=a}^{s} b_1(x-a,x)C(t-1,x-a,x), \quad t=s+2,s+3,\dots \end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
M(t,a) &= M(t-a,0)Q(a) \\
&= M^*(t-a-1)\,Q(a) + \sum_{x=a}^{t-a-2} b_2(x-a,x)C(t-a-1,x-a,x)Q(a), \\
&\qquad t=1,2,3\cdots,s+1 \qquad\qquad (3.2)\\
&= \sum_{x=a}^{s} b_{2(x-a,x)}C(t-a-1,x-a,x)Q(a), \\
&\qquad t=s+2,s+3,\cdots
\end{aligned}
$$

where $F^*(t)$ and $M^*(t)$ for t= 0, 1, 2, ...,s, express the female and male offspring from the couples formed by the ancestral individuals alive at time t. Also, the ancestral males and females are male age group x and female age group x-a at time 0 : that is M(0,x) and F(0, x-a) for x= a, a+1, . . . , s. The maximum number of couples of a given age group is equal to the minimum of the numbers of females or males in the age group : that is

$$C(t,x-a,x) = \min\{F(t,x-a),\ M(t,x)\} \quad \text{for } x=a,\ a+1,\ .\ .\ .\ ,\quad t=0,\ 1,\ .\ .\ .\ . \qquad (3.3)$$

We will take this to be the number of couples and thus, will rewrite Equations (3.1) and (3.2) by using (3.3), as

$$
\begin{aligned}
F(t,0) &= F^*(t-1) + \sum_{x=a}^{t-2} b_{1(x-a,x)}\min\{F(t,x-a),M(t,x)\}, \\
&\qquad t=1,2,\ldots,s+1 \qquad\qquad (3.4)\\
&= \sum_{x=a}^{s} b_1(x-a,x)\min\{F(t,x-a),M(t,x)\}, \\
&\qquad t=s+2,s+3,\ldots
\end{aligned}
$$

and

$$
\begin{aligned}
M(t,a) = {} & M^*(t-a-1)Q(a) \\
& + \sum_{x=a}^{t-a-2} b_2(x-a,x)\min\{F(t-a-1,x-a),M(t-a-1,x-a)\}Q(a), \\
& \quad t=1,2,\ldots,s+1 \qquad\qquad (3.5)\\
= {} & \sum_{x=a}^{s} b_2(x-a,x)\min\{F(t-a-1,x-a),M(t-a-1,x-a)\}Q(a), \\
& \quad t=s+2,s+3,\ldots\ .
\end{aligned}
$$

The females in age group x-a at time t are the females in age group 0 at time t-x+a that survive to age x-a for x-a ≤t. If x-a>t, the female were of age group x-a-t at time 0 and survived to time t. Similar reasoning holds for the males in age group x at time t: that is

$$F(t,x-a) = \begin{cases} F(t-x+a,0)P(x-a)\ , & x-a \le t \\ F(0,x-a-t)P(x-a)/P(x-a-t)\ , & x-a > t \end{cases} \qquad (3.6)$$

and

$$M(t,x) = \begin{cases} M(t-x+a,a)Q(x-a)\ , & x \le t \\ M(a,x+a-t)Q(x)/Q(x-t+a)\ , & x > t \end{cases} \quad . \qquad (3.7)$$

If we use these two equations in Equations (3.4) and (3.5), we obtain

$$F(t,0)=F^*(t-1)+\sum_{x=a}^{t-2} b_1(x-a,x)\min\{F(t-x+a-1,0)P(x-a),M(t-x+a-1,a)\}Q(x-a),\quad t=1,2,\ldots,s+1$$
$$=\sum_{x=a}^{s} b_2(x-a,x)\min\{F(t-x-1,0)P(x-a),M(t-x-1,a)q(x-a)\}Q(a),\quad t=s+2,s+3,\ldots \tag{3.8}$$

and

$$M(t,a)=M^*(t-a-1)Q(a)+\sum_{x=a}^{t-a-2} b_2(x-a,x)\min\{F(t-x-1,0)P(x-a),M(t-x-1,a)q(x-a)\}Q(a),\quad t=1,2,\ldots,s+1$$
$$=\sum_{x=a}^{s} b_2(x-a,x)\min\{F(t-x-1,0)P(x-a),M(t-x-1,a)q(x-a)\}Q(a),\quad t=s+2,s+3,\ldots \tag{3.9}$$

If the model's parameters are uniform in time, $F^*(t)$ and $M^*(t)$ are

$$F^*(t)=\sum_{x=a}^{s-t} b_1(x+t-a,x+t)\min\{F(0,x-a)P(x+t-a)/P(x-a),M(0,x)Q(x+t)/Q(x)\},\quad t=0,1,\cdots,s$$
$$=0 \quad \text{otherwise}$$

and

$$M^{*(t)}=\sum_{x=a}^{s-t} b_2(x+t-a,x+t)\min\{F(0,x-a)P(x+t-a)/P(x-a),M(0,x)Q(x+t)/Q(x)\},\quad t=0,1,\ldots,s$$
$$=0 \quad \text{otherwise.}$$

We make the assumption that

$$\frac{F(t,0)}{M(t,a)}=\frac{p}{q}$$ for all $t>0$, where p and q are known positive numbers such that p+q =1.This will ensure that the two equations become linear in the unknown s. Then,

$$\min\{F(t-x+a-1,0)P(x-a),M(t-x+a-1,a)Q(x-a)\}$$
$$=\frac{1}{p}F(t-x+a-1,0)\min\{pP(x-a),qQ(x-a)\}$$
$$=\frac{1}{q}M(t-x+a-1,a)\min\{pP(x-a),qQ(x-a)\}$$

and the Equations (3.8) and (3.9) simplify to

$$F(t,0) = \frac{1}{p}\sum_{x=a}^{s} b_1(x-a,x)F(t-x+a-1,0)\min\{pP(x-a),qQ(x-a)\} \quad t=s+2,s+3,\ldots \tag{3.10}$$

and

$$M(t,a) = \frac{Q(a)}{q}\sum_{x=a}^{s} b_2(x-a,x)M(t-x+a-1,a)\min\{pP(x-a),qQ(x-a)\}. \quad t=s+2,s+3,\ldots \tag{3.11}$$

Let b(x-a,x) = E(number of offspring a couple in female age group x-a and male age group x at time t contributes to age group 0 at time t+1).
That is , $b(x-a,x)=b_1(x-a,x)+b_2(x-a,x)$. Thus, $b_1(x-a,x)=pb(x-a,x)$ and $b_2(x-a,x)=qb(x-a,x)$, where p and q are positive numbers and p+q = 1. Therefore, renewal equations are follows by

$$\begin{aligned} F(t,0) &= F^*(t-1) + \sum_{x=a}^{t-2} b(x-a,x)F(t-x+a-1,0)\min\{pP(x-a),qQ(x-a)\}, \quad t=1,2,\ldots,s+1 \\ &= \sum_{x=a}^{s} b(x-a,x)F(t-x+a-1,0)\min\{pP(x-a),qQ(x-a)\}, \quad t=s+2,s+3,\ldots \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} M(t,a) &= M^*(t-a-1)Q(a) + Q(a)\sum_{x=a}^{t-a-2} b(x-a,x)M(t-x+a-1,0)\min\{pP(x-a),qQ(x-a)\}, \quad t=1,2,\ldots,s+1 \\ &= Q(a)\sum_{x=a}^{s} b(x-a,x)M(t-x+a-1,a)\min\{pP(x-a),qQ(x-a)\}, \quad t=s+2,s+3,\ldots \end{aligned} \tag{3.13}$$

Next, using (3.14) and (3.15) , we derive the reproductive values for the ancestral females and males.

3.2 Reproductive values

If we take the Laplace transform of Equation (3.12), we have that :

$$\sum_{t=1}^{\infty} e^{-rt}F(t,0) = \sum_{t=1}^{\infty} e^{-rt}F^*(t-1) + \sum_{x=a}^{\infty}\sum_{y=1}^{\infty} e^{-r(y+x-a+1)} b(x-a,x)F(y,0)\min\{pP(x-a),qQ(x-a)\}$$

$$\sum_{y=1}^{\infty} e^{-ry}F(y,0) = \sum_{y=1}^{\infty} e^{-ry}F^{*}(y-1) + \sum_{x=a}^{\infty} e^{-r(x-a+1)}\, b(x-a,x)\min\{pP(x-a),qQ(x-a)\} \sum_{y=1}^{\infty} e^{-ry}F(y,0)$$

Thus,

$$\sum_{y=1}^{\infty} e^{-ry}F(y,0) = \frac{\sum_{y=1}^{s+1} e^{-ry}F^{*}(y-1)}{1-\sum_{x=a}^{s} e^{-r(x-a+1)}b(x-a,x)\min\{pP(x-a),qQ(x-a)\}} .$$

If $e^{-ry} = \zeta^{y}$, then above equation is

$$\sum_{y=1}^{\infty} \zeta^{y}F(y,0) = \frac{\sum_{y=1}^{s+1} \zeta^{y}F^{*}(y-1)}{1-\sum_{x=a}^{s} \zeta^{(x-a+1)}b(x-a,x)\min\{pP(x-a),qQ(x-a)\}}$$

However, if there is no periodicity in reproduction and we let $\zeta^{-1} = \lambda_0$, then $F(y,0) \approx F\lambda_0^{y}$, as $y \to \infty$, where λ_0 is the simple dominant positive root of

$\phi(\lambda) = 1 - \sum_{x=a}^{s} \lambda^{-(x-a+1)}b(x-a,x)\min\{pP(x-a),qQ(x-a)\} = 0$ and F is a constant.

Hence, as $y \to \infty$

$$F = \frac{\sum_{y=1}^{s+1} \lambda_0^{-y}F^{*}(y-1}{1-1+\sum_{x=a}^{s}(x-a+1)\lambda_0^{-(x-a+1)}b(x-a,x)\min pP(x-a),qQ(x-a)\}}$$

$$= \frac{\sum_{y=1}^{s+1} \lambda_0^{-y}F^{*}(y-1}{\sum_{x=a}^{s}(x-a+1)\lambda_0^{-(x-a+1)}b(x-a,x)\min pP(x-a),qQ(x-a)\}} . \qquad (3.14)$$

Similarly,

$M(y,a) \approx M\lambda_0^{y}$ as $y \to \infty$,where M is also a constant, given by

$$M = \frac{\sum_{t=1}^{s+1} \lambda_0^{-y}M^{*}(y-a-1)}{\sum_{x=a}^{s}(x-a+1)\lambda_0^{-(x-a+1)}b(x-a,x)\min\{pP(x-a),qQ(x-a)\}} . \qquad (3.15)$$

Thus, there exist only one $\lambda_0 > 0$ such that $\phi(\lambda_0) = 0$ and $\lambda_0 = 1$ whenever $\phi(1) = 0$. The number of females in age group x-a and males in age group x at time t can be found

by using Equations (3.6) and (3.7).
Since

$$F(t,x-a)=F(t-x+a,0)P(x-a),\quad x-a\le t \quad \text{and}$$
$$M(t,x)=M(t-x,0)Q(x),\quad x\le t,$$

we obtain $F(t,x-a)=F(t-x+a,0)P(x-a)\approx F\lambda_0^{t-x+a}P(x-a),\quad t\to\infty$ and

$M(t,x)=M(t-x,0)Q(x)\approx M\lambda_0^{t-x}Q(x),\quad t\to\infty$ for x=a, a+1, . . . ,n .

Thus, if $\lambda_0>1$, then, as t becomes large the population of females and males grows geometrically, if $\lambda_0=1$ the population is constant in size and $\lambda_0<1$ then the population decreases. Let us now consider the possibility of the existence of a reproductive value for the ancestral females and males. Consider the constants F and M in more detail. These are defined by Equations (3.14) and (3.15).

$$F=\nu\sum_{x=1}^{s+1}\lambda_0^{-x}F^*(x-1),$$

where $\frac{1}{\nu}=\sum_{x=a}^{s}(x-a+1)\lambda_0^{-(x-a+1)}b(x-a,x)\min\{pP(x-a),qQ(x-a)\}$ and

$$F^*(x-1)=p\sum_{y=a}^{s-x+1}b(y+x-a-1,y+x-1)\min\{F(0,y-a)p(y+x-a-1)/p(y-a),M(0,y)Q(y+x-1)/Q(y)\} \tag{3.16}$$

for x=1, 2, . . . ,s+1 and $\frac{F^*(x-1)}{p}=\frac{M^*(x-a-1)}{q}$.

However, we can make a similar definition for the reproduction value of a couple if we assume that the ancestral population has equal numbers of females and males in each of the possile age groups. That is, $F(0,x-a)=M(0,x)=C(0,x-a,x)$. Then, from Equation (3.16),

$$F^*(x-1)=p\sum_{y=a}^{s-x+1}b(y+x-a-1,y+x-1)C(0,y-a,y)\min\{P(y+x-a-1)/P(y-a),Q(y+x-1)/Q(y)\}$$

for x=1,2, . . . ,s+1

and

$$\begin{aligned}F&=p\nu\sum_{x=1}^{s+1}\lambda_0^{-x}\sum_{y=a}^{s-x+1}b(y+x-a-1,y+x-1)C(0,y-a,y)\min\{P(y+x-a-1)/P(y-a),Q(y+x-1)/Q(y)\}\\&=p\nu\sum_{y=a}^{s}C(0,y-a,y)\sum_{x=a}^{s-y+1}\lambda_0^{-x}b(y+x-a-1,y+x-1)\min\{P(y+x-a-1)/P(y-a),Q(y+x-1)/Q(y)\}\end{aligned}$$

We define

$\nu_f(x-a)$ = reproductive value for an ancestral female in age x-a
= coefficient of C(0, x-a, x) in F.

$\nu_m(x)$ = reproductive value for an ancetral male in age x
= coefficient of C(0, x-a, x) in F.

Then, the reproductive values are following results.

Results :

The reproductive values for an ancestral couple in female age group x-a and male age group x with respect to the growth of females and males are, respectively.

$$\nu_f(x-a) = p\nu\lambda_0^x \sum_{y=x}^{s} \lambda_0^{-(y+1)} b(y-a,y)\min\{P(y-a)/P(x-a), Q(y)/Q(x)\} \quad \text{for } x=a,a+1,\cdots,s$$
$$= 0 \qquad \text{otherwise} \tag{3.17}$$

and

$$\nu_m(x) = q\nu\lambda_0^x \sum_{y=x}^{s} \lambda_0^{-(y+1)} b(y-a,y)\min\{P(y-a)/P(x-a), Q(y)/Q(x)\} \quad \text{for } x=a,a+1,\cdots,s \,.$$
$$= 0 \qquad \text{otherwise} \tag{3.18}$$

If x>s, we have defied $\nu_f(x-a)$ and $\nu_m(x)$ to be zero, since any individual of age greater than s does not reproduce. We also consider the concept of a total reproductive value. That is,

$$\nu_f(t) = \sum_{x=a}^{s} F(0,x-a)\nu_f(x-a), \quad t\geq 0 \quad \text{and} \quad \nu_m(t) = \sum_{x=a}^{s} M(0,x)\nu_m(x), \quad t\geq 0 \,.$$

4. Summary

We have been considering models for the growth of population with overlapping generations. We introduced a model in which couples must be formed before reproduction occurs. If the maximum number of couples is formed, and if the couples are only formed from females of age x-a and males of age x at time t (a>0), we solved the renewal equations. We have shown the females and males asymptotically grow in numbers at the same rate.. However, the parameter λ_0 which determines the rate of growth of both the females and males in the population is the solution which is larger in absolute value than any other solution to the characteristic equation . This characteristic equation is a function of the survival probabilities of the females and the males and it is also a function of the birth rates for the females and males. The concept of the reproductive values ((3.17) and (3.18)) is also valid for this model. The reproductive value of an ancestral female in age x-a and male in age x is positive as long as the couple is fertile.

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