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Bootstrapping Unified Process Capability Index

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Abstract

A family of some capability indices $\{C_p(\alpha,\beta);\alpha \geq 0, \beta \geq 0\}$, containing the indices C_p, C_{pk}, C_{pm}, and C_{pmk}, has been defined by Vännman(1993) for the case of two-sided specification interval. By varying the parameters of the family various capability indices with suitable properties are obtained. We derive the asymptotic distribution of the family $\left\{\hat{C}_p(\alpha,\beta);\alpha \geq 0, \beta \geq 0\right\}$ under general proper conditions. It is also shown that the bootstrap approximation to the distribution of the estimator $\hat{C}_p(\alpha,\beta)$ is valid for almost all sample sequences. These asymptotic distributions would be used in constructing some bootstrap confidence intervals.

Key Words : Process Capability Index; Asymptotic Distribution; Consistency of Bootstrap; Pivotal Quantity; Bootstrap Confidence Interval.

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1. INTRODUCTION

Some process capability indices(PCIs) are used to determine whether a production process is capable of producing items within a specified tolerance. They are considered as a practical tool by several advocates of statistical process control in industry. The three most widely used capability indices in industry today are

$$C_p = \frac{USL - LSL}{6\sigma}, \quad C_{pk} = \frac{\min(USL - \mu, \mu - LSL)}{3\sigma},$$

and

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}}.$$

Here USL and LSL denote the upper and lower specification limits for the process, μ and σ denote the mean and standard deviation of the process distribution, and T denotes the target value for the process. Chan, Xiong and Zhang(1990) have derived the asymptotic distributions for three estimators of capability indices mentioned above. Also, Pearn, Kotz and Johnson(1992) introduced C_{pmk}, as

$$C_{pmk} = \frac{\min(USL - \mu, \mu - LSL)}{3\sqrt{\sigma^2 + (\mu - T)^2}}$$

The family of capability indices, depending on two non-negative parameters α and β, is defined as follows(Vännman, 1993):

$$C_p(\alpha, \beta) = \frac{d - \alpha|\mu - M|}{3\sqrt{\sigma^2 + \beta(\mu - T)^2}} \tag{1.1}$$

where $d = (USL - LSL)/2$, i.e., half the length of the specification interval, $M = (USL + LSL)/2$, i.e., the mid-point of the specification interval. By varying the parameters of the family various indices with different properties can be obtained. By letting α=0 or 1 and β = 0 or 1 in unified process capability index (1.1), we obtain four basic indices mentioned above, i.e., $C_p(0,0) = C_p$, $C_p(1,0) = C_{pk}$, $C_p(0,1) = C_{pm}$, $C_p(1,1) = C_{pmk}$.

One reason for introducing capability indices like $C_p(\alpha, \beta)$ was to achieve sensitivity for departures of the process mean μ from the target value T. If it is of interest to have a capability index that is very sensitive with regard to departure of the process mean μ from the target value T, then the values of α and β should be large. See Vännman and Kotz(1995).

However, studying the estimator $\hat{C}_p(\alpha, \beta)$ of $C_p(\alpha, \beta)$ and taking its statistical properties into account, Vännman and Kotz(1995) have suggested that choosing large values of α and β is not desirable, since in that case the bias and the mean square error of the estimator could be large.

On the other hand, Efron(1979) introduced the bootstrap method which is one of the most popular methods in statistics. Bickel and Freedman(1981) have shown that the bootstrap works for means, and hence for pivotal quantities of the familiar t-statistics sort; they have made an extension to multidimensional data. Franklin and Wasserman(1991, 1992) should be regarded as the pioneers of application of bootstrap methodology in estimation of capability indices. The boostrap method achieved remarkably rapid acceptance among statistical practitioners since then. It is not until very recently that its application in the field of PCIs has been developed. Also, Nam and Park(1995) have discussed some bootstrap confidence intervals for PCIs. However, the relevance of asymptotic theory to PCIs' bootstrap applications has not been explored yet(Rodriquez; 1992).

In this article, we derived the asymptotic distribution of the family $\{\hat{C}_p(\alpha, \beta); \alpha \geq 0, \beta \geq 0\}$ under the condition that the fourth central moment about μ of the process distribution exists. It is shown that the bootstrap approximation to the distribution of the estimator $\hat{C}_p(\alpha, \beta)$ is valid for almost all sample sequences. The asymptotic distributions will be used in constructing some bootstrap confidence intervals in future study.

2. ASYMPTOTIC DISTRIBUTION

Let $X_1, X_2, \cdots, X_n$ be independent random variables with common distribution function F. Assume that F has finite mean μ and variance σ^2, both unknown. The conventional estimate for μ is the sample average, denoted here by $\bar{X}$. To analyze the sampling error in $\bar{X}$, it is customary to compute the sample variance S^2, defined as

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

By the Theorem of Chan et al.(1990), the distribution of the pivotal quantity

$$Q_n = \sqrt{n}(\hat{C}_p - C_p)/S_p$$

tends weakly to $N(0,1)$ where S_p^2 is the plug-in estimator of the capability index $\hat{C}_p$'s variance σ_p^2(See Chan et al.(1990)). So, in this situation, the asymptotics are known. More generally, there is some theoretical interest in seeing the asymptotics for the family $\left\{\hat{C}_p(\alpha,\beta); \alpha \geq 0, \beta \geq 0\right\}$.

First, consider the plug-in estimator $\hat{C}_p(\alpha,\beta)$ of unified process capability index $C_p(\alpha,\beta)$ as follows:

$$\hat{C}_p(\alpha,\beta) = \frac{d - \alpha|\bar{X} - M|}{3\sqrt{S^2 + \beta(\bar{X} - T)^2}} \tag{2.1}$$

where $d = \frac{USL-LSL}{2}$, $M = \frac{LSL+USL}{2}$. Then we obtain the following relation with some reasonable estimators:

$$\begin{aligned}
\hat{C}_p &= \frac{d}{3S} = \hat{C}_p(0,0), \\
\hat{C}_{pk} &= \frac{d - |\bar{X} - M|}{3S} = \hat{C}_p(1,0), \\
\hat{C}_{pm} &= \frac{d}{2\sqrt{S^2 + (\bar{X} - T)^2}} = \hat{C}_p(0,1), \\
\hat{C}_{pmk} &= \frac{d - |\bar{X} - M|}{2\sqrt{S^2 + (\bar{X} - T)^2}} = \hat{C}_p(1,1).
\end{aligned}$$

Theorem 1. If $\mu_4 = E(X-\mu)^4$ exists, then

$$\sqrt{n}(\hat{C}_p(\alpha,\beta) - C_p(\alpha,\beta)) \xrightarrow{d} \begin{cases} N(0,\sigma^2_{\alpha,\beta}) & \text{for } \mu < M \\ -\frac{\alpha|Y|}{3\tau_\beta} - \frac{dZ}{6\tau_\beta^3} + \frac{d\beta Y(T-\mu)}{3\tau_\beta^3} & \text{for } \mu = M \\ N(0,\sigma^{2'}_{\alpha,\beta}) & \text{for } \mu > M \end{cases}$$

$$\begin{aligned}
\text{where } \tau_\beta^2 &= \sigma^2 + \beta(\mu - T)^2, \\
\sigma^2_{\alpha,\beta} &= \mathbf{a}'\Sigma\mathbf{a}, \quad (Y,Z) \sim BN((0,0),\Sigma), \quad \sigma^{2'}_{\alpha,\beta} = \mathbf{b}'\Sigma\mathbf{b}, \\
\mathbf{a}' &= \left(\frac{\alpha}{3\tau_\beta} + \frac{\beta(T-\mu)\{d - \alpha(M-\mu)\}}{3\tau_\beta^3} \quad -\frac{d - \alpha(M-\mu)}{6\tau_\beta^3}\right), \\
\Sigma &= \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}, \quad \mu_3 = E(X-\mu)^3, \\
\mathbf{b}' &= \left(-\frac{\alpha}{3\tau_\beta} + \frac{\beta(T-\mu)\{d + \alpha(M-\mu)\}}{3\tau_\beta^3} \quad -\frac{d + \alpha(M-\mu)}{6\tau_\beta^3}\right).
\end{aligned}$$

Proof. The proof is obtained by applying Chan et al.(1990) and considering the function $g(u,v)$ as follows :

$$g(u,v) = \frac{d - \alpha\sqrt{(u-M)^2}}{3\sqrt{v + \beta(u-T)^2}} \tag{2.2}$$

Of course, the result in case of $\mu = M$ is calculated with some limit theorems.

Corollary 1 (Chan et al. (1990).

(a) $\sqrt{n}(\hat{C}_p - C_p) \xrightarrow{d} N(0, \sigma_p^2)$, where $\sigma_p^2 = \dfrac{(\mu_4 - \sigma^4)d^2}{36\sigma^6}$

(b) $$\sqrt{n}(\hat{C}_{pk} - C_{pk}) \xrightarrow{d} \begin{cases} N(0, \sigma_{pk}^2), & \mu < M \\ -\frac{|Y|}{3\sigma} - \frac{dZ}{6\sigma^6}, & \mu = M \\ N(0, \sigma_{pk}^{2'}), & \mu > M \end{cases}$$

where $$\sigma_{pk}^2 = \frac{1}{9} - \frac{\mu_3\{d - (M-\mu)\}}{9\sigma^4} + \frac{(\mu_4 - \sigma^4)\{d - (M-\mu)\}^2}{36\sigma^6}$$

$$\sigma_{pk}^{2'} = \frac{1}{9} + \frac{\mu_3\{d + (M-\mu)\}}{9\sigma^4} + \frac{(\mu_4 - \sigma^4)\{d + (M-\mu)\}^2}{36\sigma^6}$$

(c) $\sqrt{n}(\hat{C}_{pm} - C_{pm}) \xrightarrow{d} N(0, \sigma_{pm}^2)$,

where $$\sigma_{pm}^2 = \frac{d^2\left\{\sigma^2(\mu-T)^2 - \mu_3(\mu-T) + \frac{1}{4}(\mu_4 - \sigma^4)\right\}}{9\tau^6}$$

$$\tau^2 = \sigma^2 + (\mu - T)^2$$

Corollary 2.

$$\sqrt{n}(\hat{C}_{pmk} - C_{pmk}) \xrightarrow{d} \begin{cases} N(0, \sigma_{pmk}^2) & \text{for } \mu < M \\ -\frac{|Y|}{3\tau} - \frac{dZ}{6\tau^3} + \frac{dY(T-\mu)}{3\tau^3} & \text{for } \mu = M \\ N(0, \sigma_{pmk}^{2'}) & \text{for } \mu > M \end{cases}$$

where

$$\begin{aligned} \sigma_{pmk}^2 &= \frac{1}{9\tau^6}\left[[\tau^2 + (T-\mu)(d-M+\mu)]^2\sigma^2 - \mu_3(d-M+\mu)[\tau^2 + \right. \\ &\quad \left. (d-M+\mu)(T-\mu)] + \frac{1}{4}(\mu_4 - \sigma^4)(d-M+\mu)^2\right], \\ (Y,Z) &\sim BN((0,0),\Sigma) \\ \sigma_{pmk}^{2'} &= \frac{1}{9\tau^6}\left[[\tau^2 + (\mu-T)(d-\mu+M)]^2\sigma^2 + \mu_3(d-\mu+M)[\tau^2 + \right. \\ &\quad \left. (d-\mu+M)(\mu-T)] + \frac{1}{4}(\mu_4 - \sigma^4)(d-\mu+M)^2\right]. \end{aligned}$$

3. BOOTSTRAP APPROXIMATION

3.1. Bootstrap Algorithm

Efron(1979) discusses the method of bootstrap for setting confidence intervals and estimating significance levels. This method consists of approximating the distribution of a function of the observations and the underlying distribution, such as a pivot, by what Efron calls the bootstrap distribution of this quantity. This distribution is obtained by replacing the unknown distribution by the empirical distribution of the data in the definition of the statistical function, and then resampling the data to obtain a Monte Carlo distribution for the resulting random variable. This method would probably be used in practice only when the distributions could not be estimated analytically. However, it is of some interest to check that the bootstrap approximation is valid in situations which are simple enough to handle analytically. Efron gives a series of examples in which this principle works, and establishes the validity of the approach for a general class of statistics when the sample space is finite.

Let F_n be the empirical distribution of $X_1, X_2, \cdots, X_n$, putting mass $1/n$ on each X_i. The bootstrap algorithm goes as follows:

- Step 1 : Given $\chi_n = (X_1, X_2, \cdots, X_n)$, let $X_1^*, X_2^*, \cdots, X_m^*$ be conditionally independent, with common distribution F_n.

- Step 2 : From the bootstrap sample $X_1^*, X_2^*, \cdots, X_m^*$, compute the sample mean $\bar{X}^*$ and sample variance S^{*2}.

$$\begin{aligned} \bar{X}^* &= \frac{1}{m}\sum_{i=1}^{m} X_i^*, \\ S^{*2} &= \frac{1}{m-1}\sum_{i=1}^{m}(X_i^* - \bar{X}^*)^2 \end{aligned}$$

- Step 3 : Compute the bootstrap plug-in estimator of unified process capability index $C_p(\alpha, \beta)$ as follows:

$$\hat{C}_p^*(\alpha,\beta) = \frac{d - \alpha|\bar{X}^* - M|}{3\sqrt{S^{2*} + \beta(\bar{X}^* - T)^2}} \tag{3.1}$$

We obtain four bootstrap estimators from (3.1).

$$\hat{C}_p^* = \frac{d}{3S^*} = \hat{C}_p^*(0,0)$$

$$\hat{C}^*_{pk} = \frac{d - |\bar{X}^* - M|}{3S^*} = \hat{C}^*_p(1,0)$$
$$\hat{C}^*_{pm} = \frac{d}{3\sqrt{S^{*2} + (\bar{X}^* - T)^2}} = \hat{C}^*_p(0,1)$$
$$\hat{C}^*_{pmk} = \frac{d - |\bar{X}^* - M|}{3\sqrt{S^{*2} + (\bar{X}^* - T)^2}} = \hat{C}^*_p(1,1)$$

3.2. Asymptotic Bootstrap Distribution

We allow the resample size m to differ from the number n of data points, to estimate the distribution of the bootstrap pivotal quantity, say, $Q^*_m = \sqrt{m}(\hat{C}^*_p - \hat{C}_p)/S^*_p$, where S^{*2}_p is the bootstrap version of the plug-in estimator S^2_p of the variance σ^2_p.

In the resampling, the n data points $X_1, X_2, \cdots, X_n$ are treated as a population, with distribution function F_n and mean $\bar{X}$; and $\bar{X}^*$ is considered as an estimator of $\bar{X}$. First, take $m = n$. The idea is that the behavior of the bootstrap pivotal quantity Q^*_n mimics that of Q_n. Thus, the distribution of Q^*_n could be computed from the data and used to approximate the unknown sampling distribution of Q_n. Or even more directly, the bootstrap distribution of $\sqrt{n}(\hat{C}^*_p - \hat{C}_p)$ could be used to approximate the sampling distribution of $\sqrt{n}(\hat{C}_p - C_p)$. Either approach would be lead to confidence intervals for C_p, and would be useful if the bootstrap approximation were valid.

More generally, we have an interest in constructing some bootstrap confidence intervals for the index $C_p(\alpha,\beta)$. That is, we show that the bootstrap approximation to the distribution of the estimator $\hat{C}_p(\alpha,\beta)$ is valid for almost all sample sequences.

Lemma 1. Along almost all sample sequences given $\mathcal{X}_n = (X_1, X_2, \cdots, X_n)$, as m and n tend to ∞:

$$\sqrt{m}(\bar{X}^* - \bar{X}, S^{*2} - S^2)|\mathcal{X}_n \xrightarrow{d} BN\left((0,0), \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}\right)$$

Proof. Let F_n be the empirical distribution of $\binom{X_1}{X_1^2}\binom{X_2}{X_2^2}, \cdots, \binom{X_n}{X_n^2}$. Given $\binom{X_1}{X_1^2}\binom{X_2}{X_2^2}, \cdots, \binom{X_n}{X_n^2}$, let $\binom{X_1^*}{X_1^{*2}}\binom{X_2^*}{X_2^{*2}}, \cdots, \binom{X_m^*}{X_m^{*2}}$ be conditionally independent, with common distribution F_n.

With Bickel and Freedman(1981:Theorem 1 and Theorem 2) and Mallows(1972), we obtain the following limiting distribution. As n and m tend to ∞:

$$\sqrt{m}\left(\bar{X}^* - \bar{X}, \frac{1}{m}\sum_{i=1}^{m} X_i^{*2} - \frac{1}{n}\sum_{i=1}^{n} X_i^2\right)\Big|\mathcal{X}_n$$

$$\xrightarrow{d} BN\left((0,0), \begin{pmatrix} \sigma^2 & \mu_3 + 2\mu\sigma^2 \\ \mu_3 + 2\mu\sigma^2 & \mu_4 + 4\mu\mu_3 + 4\mu^2\sigma^2 - \sigma^4 \end{pmatrix}\right) \tag{3.2}$$

Hence, we obtain Lemma 1 by (3.2) and simple calculations.

Lemma 2. Assume that function $g(u,v)$ is differentiable. Along almost all sample sequences given $\mathcal{X}_n = (X_1, X_2, \cdots, X_n)$, as m and n tend to ∞:

$$\sqrt{m}\left(g(\bar{X}^*, S^{*2}) - g(\bar{X}, S^2)\right)|\mathcal{X}_n \xrightarrow{d} N(0, D'\Sigma D)$$

where $D' = \begin{pmatrix} \frac{\partial g(u,v)}{\partial u}|_{\mu,\sigma^2} & \frac{\partial g(u,v)}{\partial v}|_{\mu,\sigma^2} \end{pmatrix} \neq (0,0)$.

Proof. The Lemma 2 follows from Lemma 1 and the Theorem A(p.122) of Serfling(1980).

Theorem 2. Along almost all sample sequences given $\mathcal{X}_n = (X_1, X_2, \cdots, X_n)$, as m and n tend to ∞:

$$\sqrt{m}(\hat{C}_p^*(\alpha,\beta) - \hat{C}_p(\alpha,\beta))|\mathcal{X}_n \xrightarrow{d} \begin{cases} N(0, \sigma^2_{\alpha,\beta}) & \text{for } \mu < M \\ -\frac{\alpha|Y|}{3\tau_\beta} - \frac{dZ}{6\tau_\beta^3} + \frac{d\beta Y(T-\mu)}{3\tau_\beta^3} & \text{for } \mu = M \\ N(0, \sigma^{2'}_{\alpha,\beta}) & \text{for } \mu > M \end{cases}$$

under the same conditions as Theorem 1.

Proof. The proof is obtained by applying Lemma 1 and Lemma 2 in case of $\mu < M$ or $\mu > M$. Also, the result of the case $\mu = M$ is derived by the following calculations with some limit theorems containing the Slutsky's theorem. We consider the case $\mu = M$ as follows:

$$\begin{aligned} &\sqrt{m}\left(\hat{C}_p^*(\alpha,\beta) - \hat{C}_p(\alpha,\beta)\right)\Big|\chi_n \\ = \quad &\sqrt{m}\left(\frac{d-\alpha|\bar{X}^*-\mu|}{3\sqrt{S^{*2}+\beta(\bar{X}^*-T)^2}} - \frac{d-\alpha|\bar{X}-\mu|}{3\sqrt{S^2+\beta(\bar{X}-T)^2}}\right)|\chi_n \\ = \quad &\sqrt{m}\left(\frac{d}{3\sqrt{S^{*2}+\beta(\bar{X}^*-T)^2}} - \frac{d}{3\sqrt{S^2+\beta(\bar{X}-T)^2}}\right)|\chi_n \\ &-\alpha\sqrt{m}\left(\frac{|\bar{X}^*-\mu|}{3\sqrt{S^{*2}+\beta(\bar{X}^*-T)^2}} - \frac{|\bar{X}-\mu|}{3\sqrt{S^2+\beta(\bar{X}-T)^2}}\right)|\chi_n \end{aligned}$$

The first term is calculated as follows:

$$\sqrt{m}\left(\frac{d}{3\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}}-\frac{d}{3\sqrt{S^{2}+\beta(\bar{X}-T)^{2}}}\right)\Bigg|\mathcal{X}_{n}$$

$$=\frac{-\sqrt{m}d\left(\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}-\sqrt{S^{2}+\beta(\bar{X}-T)^{2}}\right)}{3\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}\sqrt{S^{2}+\beta(\bar{X}-T)^{2}}}\Bigg|\mathcal{X}_{n}$$

$$=\frac{-\sqrt{m}d[S^{*2}-S^{2}+\beta(\bar{X}^{*}-T+\bar{X}-T)(\bar{X}^{*}-\bar{X})]}{3\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}\sqrt{S^{2}+\beta(\bar{X}-T)^{2}}(\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}+\sqrt{S^{2}+\beta(\bar{X}-T)^{2}})}\Bigg|\mathcal{X}_{n}$$

$$\xrightarrow{d}\ -\frac{dZ}{6\tau_{\beta}^{3}}+\frac{d\beta(T-\mu)Y}{3\tau_{\beta}^{3}}\qquad \text{as } m\to\infty \text{ and } n\to\infty \tag{3.3}$$

where $(Y,Z)\sim BN((0,0),\Sigma),\ \tau_{\beta}^{2}=\sigma^{2}+\beta(\mu-T)^{2}$

Also, the second term can be calculated as follows:

$$-\alpha\sqrt{m}\left(\frac{|\bar{X}^{*}-\mu|}{3\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}}-\frac{|\bar{X}-\mu|}{3\sqrt{S^{2}+\beta(\bar{X}-T)^{2}}}\right)\Bigg|\mathcal{X}_{n}$$

$$=\frac{-\alpha\sqrt{m}\left[|\bar{X}^{*}-\mu|\sqrt{S^{2}+\beta(\bar{X}-T)^{2}}-|\bar{X}-\mu|\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}\right]}{3\sqrt{S^{*2}+\beta(\bar{X}^{*}-T)^{2}}\sqrt{S^{2}+\beta(\bar{X}-T)^{2}}}\Bigg|\mathcal{X}_{n}$$

$$\xrightarrow{d}\ -\frac{\alpha|Y|}{3\tau_{\beta}}\qquad \text{as } m\to\infty \text{ and } n\to\infty \tag{3.4}$$

by rationalizing the numerator and applying some limit theorems to it. Two results (3.3) and (3.4) imply Theorem 2 for the case $\mu = M$ immediately. This completes the proof. Of course, these limiting distributions are identical with those of Theorem1.

Corollary 3 (Kim and Cho(1995)). Along almost all sample sequences given $\mathcal{X}_n = (X_1, X_2, \cdots, X_n)$, as n and m tend to ∞:

$$(a)\qquad \sqrt{m}(\hat{C}_p^{*}-\hat{C}_p)|\mathcal{X}_n \xrightarrow{d} N(0,\sigma_p^2)$$

$$(b)\qquad \sqrt{m}(\hat{C}_{pk}^{*}-\hat{C}_{pk})|\mathcal{X}_n \xrightarrow{d} \begin{cases} N(0,\sigma_{pk}^2) & \text{for } \mu<M \\ -\frac{|Y|}{3\sigma}-\frac{dZ}{6\sigma^3} & \text{for } \mu=M \\ N(0,\sigma_{pk}^{2'}) & \text{for } \mu>M \end{cases}$$

$$(c)\qquad \sqrt{m}(\hat{C}_{pm}^{*}-\hat{C}_{pm})|\mathcal{X}_n \xrightarrow{d} N(0,\sigma_{pm}^2)$$

Corollary 4. Along almost all sample sequences given $\mathcal{X}_n = (X_1, X_2, \cdots, X_n)$, as m and n tend to ∞:

$$\sqrt{m}(\hat{C}^*_{pmk} - \hat{C}_{pmk})|\mathcal{X}_n \xrightarrow{d} \begin{cases} N(0, \sigma^2_{pmk}) & \text{for } \mu < M \\ -\frac{|Y|}{3\tau} - \frac{dZ}{6\tau^3} + \frac{dY(T-\mu)}{3\tau^3} & \text{for } \mu = M \\ N(0, \sigma^{2'}_{pmk}) & \text{for } \mu > M \end{cases}$$

Since the variance σ^2_p is a continuous function of σ^2 and μ_4, we can easily show the following result.

Remark. Along almost all sample sequences given $\mathcal{X}_n = (X_1, X_2, \cdots, X_n)$, as n and m tend to ∞:

$S^*_p \to \sigma_p$ in conditional probability: that is, for arbitrary positive ϵ as m and n tend to ∞,

$$P\left\{|S^*_p - \sigma_p| > \epsilon \middle| \mathcal{X}_n\right\} \to 0 \text{ a.s.}$$

For other cases, similar results hold.

4. FURTHER TOPICS

The consistency of the bootstrap(Theorem 1 and Theorem 2) guarantees that some approximate statistical inferences for the family of unified process capability index $\{C_p(\alpha, \beta); \alpha \geq 0, \beta \geq 0\}$, containing the indices C_p, C_{pk}, and C_{pmk}, can be performed.

In particular, these capability indices are widely used to estimate whether a process is capable. Recently, techniques and tables were developed to construct lower 95% confidence limits for each index. These techniques assume the underlying process is normally distributed. But some processes that are modestly nonnormal do occur and can be hard to detect. Therefore, some nonparametric bootstrap lower confidence limits(Standard bootstrap, Percentile bootstrap, Bias-corrected percentile bootstrap, Studentized bootstrap, Hybrid bootstrap, Backward bootstrap, Bias-corrected bootstrap, Accelerated bias-corrected bootstrap, Coverage-corrected percentile bootstrap(Hall (1986, 1988), Franklin and Wasserman(1992)) will be compared for each of these capability indices. Then we will choose better nonparametric bootstrap confidence intervals for these capability indices.

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