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Existence Condition for the Stationary Ergodic New Laplace Autoregressive Model of Order p - NLAR(p)

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Abstract

The new Laplace autoregressive model of order 2 - NLAR(2) studied by Dewald and Lewis (1985) is extended to the p-th order model - NLAR(p). A necessary and sufficient condition for the existence of an innovation sequence and a stationary ergodic NLAR(p) model is obtained. It is shown that the distribution of the innovation sequence is given by the probabilistic mixture of independent Laplace distributions and a degenrate distribution.

Key Words : New laplace autoregressive model; Stationary process; Ergodic process; Laplace distribution; Degenerate distribution.

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1. INTRODUCTION

It is usually assumed in time series analysis that the marginal distribution of $\{X_n\}$ is Gaussian. However, Gaussian distribution is not always approprite. In recent years, a number of non-Gaussian time series models have been developed. The need for such models arises from the fact that the exponential, Gamma, and Laplace distribution are more appropriate for modelling highly skewed and long tailed series than Gaussian distribution.

Lawrance and Lewis(1981, 1985) suggested a new exponential autoregressive time series model - NEAR(1) and NEAR(2) for modelling positive and highly skewed data, e.g., wind speed, service time in a queue, and daily flows of a river. Chan(1988) extended the NEAR(2) model to the p-th order NEAR model and obtained the necessary and sufficient condition for the existence of the stationary ergodic NEAR(p) model.

The important property of the NEAR model is simple and analytically tractable. But this simplicity is bought at the price of the autocorrelations being non-negative.

Dewald and Lewis(1985) proposed a new Laplace autoregresive time series model - NLAR(2) for modelling large kurtosis and long tailed data, e.g., position error, response rate, and speech waves.

The NLAR(2) model provides great flexibility in the sense of the broad range of autocorrelations and partially time reversible of third order moments.

In this paper, the NLAR(2) model is generalized to the NLAR(p) model. The detailed explanation of the NLAR(2) model and its extension are discussed in Section 2. Necessary and sufficient condition for the existence of a stationary ergodic NLAR(p) model and an innovation sequence are derived in Section 3.

2. PRELIMINARIES

The NLAR(2) model takes the form

$$X_n = e_n + \begin{cases} \beta_1 X_{n-1} & w.p. \quad \alpha_1 \\ \beta_2 X_{n-2} & w.p. \quad \alpha_2 \\ 0 & w.p. \quad \alpha_0 = 1 - \alpha_1 - \alpha_2 \end{cases} , \quad n = 1, 2, 3, \cdots \tag{2.1}$$

where the distribution of the i.i.d. innovation sequence $\{e_n\}$ is chosen so that the marginal distribution of the stationary sequence $\{X_n\}$ is standard Laplace, i.e.,

$$f(x) = \frac{1}{2}exp(-|x|), \quad -\infty < x < \infty. \tag{2.2}$$

Dewald and Lewis(1985) showed that there exists a unique strictly stationary and ergodic solution to the equation (2.1) if and only if $\alpha_1\beta_1^2 + \alpha_2\beta_2^2 < 1$. They also showed that if $0 < \alpha_i < 1$, $0 < |\beta_i| < 1$, $i = 1, 2$, $\alpha_1 + \alpha_2 < 1$, then e_n is given by

$$e_n = \begin{cases} L_n & w.p. \quad 1 - p_2 - p_3 \\ |b_2|L_n & w.p. \quad p_2 \\ |b_3|L_n & w.p. \quad p_3 \end{cases} \tag{2.3}$$

where $\{L_n\}$ is i.i.d. standard Laplace variable and is independent of $\{X_n\}$. The values of b_2, b_3, p_2, and p_3 are given by

$$p_2 = \frac{\alpha_1\beta_1^2(b_2^2 - \beta_2^2) + \alpha_2\beta_2^2(b_2^2 - \beta_1^2)}{(b_2^2 - b_3^2)(1 - b_2^2)} \tag{2.4}$$

$$p_3 = -\frac{\alpha_1\beta_1^2(b_3^2 - \beta_2^2) + \alpha_2\beta_2^2(b_3^2 - \beta_1^2)}{(b_2^2 - b_3^2)(1 - b_3^2)} \tag{2.5}$$

$$1 > b_2^2 = \frac{1}{2}\left\{s + (s^2 - 4r)^{\frac{1}{2}}\right\} > b_3^2 = \frac{1}{2}\left\{s - (s^2 - 4r)^{\frac{1}{2}}\right\} > 0 \tag{2.6}$$

$$s = (1 - \alpha_1)\beta_1^2 + (1 - \alpha_2)\beta_2^2 \tag{2.7}$$

$$r = (1 - \alpha_1 - \alpha_2)\beta_1^2\beta_2^2 \tag{2.8}$$

The NLAR(p) model can be constructed analogously to the equation (2.1) and hence takes the form

$$X_n = e_n + \begin{cases} 0 & w.p. \quad \alpha_0 = 1 - \alpha_1 - \cdots - \alpha_p \\ \beta_1 X_{n-1} & w.p. \quad \alpha_1 \\ \beta_2 X_{n-2} & w.p. \quad \alpha_2 \\ \vdots & \\ \beta_p X_{n-p} & w.p. \quad \alpha_p \end{cases}, \quad n = 1, 2, 3, \cdots \tag{2.9}$$

where the distribution of the i.i.d. innovation sequence $\{e_n\}$ is chosen so that the marginal distribution of the stationary sequence $\{X_n\}$ is standard Laplace.

The equation (2.9) is exactly same as that of the NEAR(p) model. The only difference is that the marginal distribution of the stationary sequence $\{X_n\}$ of NEAR(p) model is exponential with mean one. Hence the existence condition for the stationary ergodic NLAR(p) model can be obtained by using the existence condition for the stationary ergodic NEAR(p) model.

Let Z denote the set of integers and let $\{J(n), n \in Z\}$ and $\{e_n, n \in Z\}$ be two sequences of i.i.d. random variables where $\{J(n)\}$ and $\{e_n\}$ are assumed to be independent. It is further assumed that $J(n)$ is discrete random variable distributed as follows;

$$J(n) = \begin{cases} 0 & w.p. \quad \alpha_0 = 1 - \alpha_1 - \quad - \alpha_p \\ 1 & w.p. \quad \alpha_1 \\ 2 & w.p. \quad \alpha_2 \\ \vdots & \\ p & w.p. \quad \alpha_p \end{cases} \tag{2.10}$$

Let $\beta_{J(n)}$ be a discrete random variable taking constants $\beta_1, \beta_2, \cdots, \beta_p$ in $J(n)$, i.e.,

$$\beta_{J(n)} = \begin{cases} 0 & w.p. \quad \alpha_0 = 1 - \alpha_1 - \quad - \alpha_p \\ \beta_1 & w.p. \quad \alpha_1 \\ \beta_2 & w.p. \quad \alpha_2 \\ \vdots & \\ \beta_p & w.p. \quad \alpha_p \end{cases} \tag{2.11}$$

Using these notations, we can express the equation (2.9) as a recursive form

$$X_n = \beta_{J(n)} X_{n-J(n)} + e_n, \quad n \in Z \tag{2.12}$$

where $J(n)$ is independent of $X_{n-1}, X_{n-2}, \cdots$.

Now, we state a theorem that the equation (2.9) admits a stationary ergodic solution.

Theorem 1. If either of the conditions

(1) $P(\beta_{J(n)} = 0) > 0$

or

(2) $P(|\beta_{J(n)}| > 0) = 1, \quad E|e_n| < \infty$, and $E|\beta_{J(n)}| < 1$

holds, then the equation (2.9) admits a strictly stationary ergodic solution X_n.

Proof . Proof closely follows the one in Chan(1988) except that $|\beta_{J(n)}|$ is substituted for $\beta_{J(n)}$.

3. EXISTENCE CONDITION OF THE STATIONARY ERGODIC NLAR(p) MODEL

We state two lemmas before we derive the main result of this paper.

Lemma 1. Let $|\beta_1|, |\beta_2|, \cdots, |\beta_p|$ be positive distinct numbers in descending order and let $\alpha_0, \alpha_1, \cdots, \alpha_p$ be positive probabilities less than one with $\sum_{i=0}^{p} \alpha_i = 1$. Define $q(t)$ for real value of t as follows.

$$q(t) = \alpha_0 h(t) + \sum_{j=1}^{p} \alpha_j \prod_{\substack{i=1 \\ i \neq j}}^{p} (1 - \beta_j^2 t^2) \tag{3.1}$$

where $h(t) = \prod_{i=1}^{p}(1 - \beta_i^2 t^2)$.

Then, for some $b_1, b_2, \cdots, b_p$ such that $|\beta_i| > |b_i| > |\beta_{i+1}|$, $i = 1, 2, \cdots, p-1$ and $|\beta_p| > |b_p|$, we have

$$q(t) = \prod_{i=1}^{p}(1 - b_i^2 t^2) \tag{3.2}$$

Proof. For the case of $p = 1$, the equation (3.1) becomes

$$q(t) = 1 - \alpha_0 \beta_1^2 t^2$$

Taking $b_1^2 = \alpha_0 \beta_1^2$ gives $|\beta_1| > |b_1| > 0$.

For the case of $p > 1$, the equation (3.1) becomes

$$q\left(\frac{1}{|\beta_i|}\right) = \frac{\alpha_i}{(\beta_i^2)^{p-1}} \prod_{\substack{i=1 \\ j \neq i}} (\beta_i^2 - \beta_j^2), \quad i = 1, 2, \cdots, p-1 \tag{3.3}$$

Since $|\beta_i|$'s are positive distinct number in descending order, the sign of $q\left(\frac{1}{|\beta_i|}\right)$ is $(-1)^{i-1}$ and the sign of $q\left(\frac{1}{|\beta_i|}\right) q\left(\frac{1}{|\beta_{i+1}|}\right)$ is negative. This implies that $q(t)$ changes sign in an interval $\left(\frac{1}{|\beta_i|}, \frac{1}{|\beta_{i+1}|}\right)$. Therefore, there exists $|b_i|$

such that $q\left(\frac{1}{|b_i|}\right) = 0$ in the interval $\left(\frac{1}{|\beta_i|}, \frac{1}{|\beta_{i+1}|}\right)$. Hence, we have $|\beta_i| > |b_i| > |\beta_{i+1}|$, $i = 1, 2, \cdots, p-1$.

Similarly, the sign of $q\left(\frac{1}{|\beta_p|}\right)$ is $(-1)^{p-1}$ and the sign of $q(t)$ is $(-1)^p$ for $t \to \pm\infty$. This implies that $q(t)$ changes sign in $\left(\frac{1}{\beta_p}, \infty\right)$ for $\beta_p > 0$, and in $\left(-\infty, \frac{1}{\beta_p}\right)$ for $\beta_p < 0$. Therefore, there exsits b_p such that $q\left(\frac{1}{b_i}\right) = 0$ in $\left(\frac{1}{\beta_p}, \infty\right)$ for $\beta_p > 0$, and in $\left(-\infty, \frac{1}{\beta_p}\right)$ for $\beta_p < 0$. Hence, we have $|\beta_p| > |b_p| > 0$.

The proof is completed from the fact that $q(t)$ is the p-th order of polynomial in t^2 and its roots are $\frac{1}{|b_1|}, \frac{1}{|b_2|}, \cdots, \frac{1}{|b_p|}$.

Lemma 2. Assume that the conditions of the Lemma 1 hold and further assume that $\alpha_0 = 0$. Then, for some $b_1, b_2, \cdots, b_{p-1}$ such that $|\beta_i| > |b_i| > |\beta_{i+1}|$, $i = 1, 2, \cdots, p-1$, we have

$$q(t) = \prod_{i=1}^{p-1} (1 - b_i^2 t^2) \tag{3.4}$$

Proof. Proof is omitted since it is similar as that of the Lemma 1.

Now we state the main theorem of this paper.

Theorm 2. There exist an innovation sequense $\{e_n\}$ such that the equation (2.9) admits a stationary ergodic solution X_n with marginal standard Laplace distribution if and only if

$$0 \le |\beta_{J(n)}| \le 1 \quad \text{and} \quad E|\beta_{J(n)}| < 1 \tag{3.5}$$

Proof. It is clear from the Theorm 1 that necessary condition holds.

To prove sufficiency, let $\varphi_X(t)$ and $\varphi_e(t)$ be the moment generating functions of X_n and e_n respectively.

Assume that X_n is stationary and distributed marginal standard Lalpace. Then, we have that for $|t| < \frac{1}{|\beta_i|}$, $i = 1, 2, \cdots, p$,

$$\varphi_X(t) = \varphi_e(t)\{\alpha_0 + \alpha_1\varphi_X(\beta_1 t) + \alpha_2\varphi_X(\beta_2 t) + \cdots + \alpha_p\varphi_X(\beta_p t)\} \tag{3.6}$$

and

$$\varphi_e(t) = \left\{(1-t^2)(\alpha_0 + \frac{\alpha_1}{1-\beta_1^2 t^2} + \frac{\alpha_2}{1-\beta_2^2 t^2} + \cdots + \frac{\alpha_p}{1-\beta_p^2 t^2}\right\}^{-1} \tag{3.7}$$

Now, we will find the condition for the right hand side of the equation (3.7) to be a valid moment generating function.

If $\beta_{J(n)} = 0$, then $\varphi_e(t)$ is trivially the moment generating function of the standard Laplace distribution.

For the case of $|\beta_{J(n)}| \neq 0$, we can assume without loss of generality that $|\beta_1|, |\beta_2|, \cdots, |\beta_p|$ are positive distinct numbers in descending order and $\alpha_1, \cdots, \alpha_p$ are positive.

Case 1 : $0 < \alpha_0 < 1$.

It is seen from the Lemma 2 that $\varphi_e(t)$ is given by

$$\varphi_e(t) = \frac{\prod_{i=1}^{p}(1-\beta_i^2 t^2)}{(1-t^2)\prod_{i=1}^{p}(1-b_i^2 t^2)} \tag{3.8}$$

By partial fraction expansion, the right hand side of the equation (3.8) can be expressed as

$$\frac{a_1}{1-b_1^2 t^2} + \frac{a_2}{1-b_2^2 t^2} + \cdots + \frac{a_p}{1-b_p^2 t^2} + \frac{a_{p+1}}{1-t^2} \tag{3.9}$$

where the coefficients a_k, $k = 1, 2, \ldots, p, p+1$ are given by

$$a_k = \begin{cases} \frac{\prod_{i=1}^{p}(b_k^2-\beta_i^2)}{(b_k^2-1)\prod_{i\neq k}^{p}(b_k^2-b_i^2)} & k = 1, 2, \cdots, p, \quad |\beta_j| \neq 1, b_k \neq 1 \\ \frac{\prod_{i\neq j}^{p}(b_k^2-\beta_i^2)}{\prod_{i\neq k}^{p}(b_k^2-b_i^2)} & k = 1, 2, \cdots, p, \quad |\beta_j| = 1 \end{cases} \tag{3.10}$$

$$a_{p+1} = \begin{cases} \frac{\prod_{i=1}^{p}(1-\beta_i^2)}{\prod_{i=1}^{p}(1-b_i^2)} & |\beta_j| \neq 1, b_i \neq 1 \\ 0 & |\beta_j| = 1 \end{cases} \tag{3.11}$$

Let $F(x)$ denote the distribution function of e_n. Inverting (3.9), we then have

$$F(x) = \int_{-\infty}^{x} \left\{ \sum_{i=1}^{p} \frac{a_i}{2|b_i|}\, exp\left(-\frac{|t|}{|b_i|}\right) + \frac{a_{p+1}}{2} exp(-|t|) \right\} dt \tag{3.12}$$

In order that $\varphi_e(t)$ is a moment generating function of some random variable, $F(x)$ should be non-decreasing, continuous from right, and unity at infinity.

It is straigtforward that $F(x)$ attains unity at infinity, i.e.,

$$F(\infty) = \varphi_e(0) = 1$$

Continuity from right is clear. The non-decreasing property of $F(x)$ is equivalent to the non-negativity of the integrand which is satisfied from the eqution (3.10) and (3.11) if $0 \leq |\beta_1|, |\beta_2|, \cdots, |\beta_p| \leq 1$.

Case 2 : $\alpha_0 = 0$.

For the case of $p = 1$, we have

$$\varphi_e(t) = \beta_1^2 + \frac{1-\beta_1^2}{1-t^2} \tag{3.13}$$

Inverting the equation (3.13), we have

$$F(x) = \int_{-\infty}^{x} \left\{ \beta_1^2 g(t) + \frac{1-\beta_1^2}{2} exp(-|t|) \right\} dt \tag{3.14}$$

where $g(t)$ is degerate at zero.

It is easily seen that the equation (3.14) is a distribution function if and only if $|\beta_1| \leq 1$.

For the case of $p > 1$, $q(t)$ takes the form of the equation (3.4).

By partial fraction, we have

$$\varphi_e(t) = c + \frac{a_1}{1-b_1^2t^2} + \frac{a_2}{1-b_2^2t^2} + \cdots + \frac{a_{p-1}}{1-b_{p-1}^2t^2} + \frac{a_{p+1}}{1-t^2} \tag{3.15}$$

where $c = \prod_{i=1}^{p} \beta_i^2 / \prod_{i=1}^{p-1} b_i^2$ and a_k's are given in the equation (3.10) and (3.11) respectively except that the subscripts run from 1 to $p-1$.

Inverting (3.15), we have

$$F(x) = c + \int_{-\infty}^{x} \left\{ \sum_{i=1}^{p-1} \frac{a_i}{2|b_i|} exp\left(\frac{-|t|}{|b_i|}\right) + \frac{a_p}{2} exp(-|t|) \right\} dt \tag{3.16}$$

Similarly as in the case 1, it can be shown that $F(x)$ is a proper distribution of e_n if and only if $0 \leq |\beta_1|, |\beta_2|, \cdots, |\beta_p| \leq 1$.

This completes the proof of the Theorem 2.

Remark 1. For the case of $p = 2$, the existence condition of the above theorem reduces that in Dewald and Lewis(1985).

The distribution of the innovation seqence $\{e_n\}$ can be obtained from the proof of the Teorem 2 and is summarized in the following corollary.

Corollary 1. Assume that the condition in the Teorem 2 holds and further assume that α_i's are positive and $0 < |\beta_i| < 1$, $i = 1, 2, \cdots, p$. Then the distribution of the innovation seqence $\{e_n\}$ is given by a probabilistic mixture of independent Laplace distribution. If $\alpha_0 = 0$, then one of the mixture is degenerate at zero.

Proof. This follows from the equation (3.12) and (3.14).

Remark 2. It can be shown that the distribution of $\{e_n\}$ for the case of $p = 2$ reduces to the equation (2.3).

4. CONCLUDING REMARKS

In this paper, the NLAR(2) model developed for the variables with large kurtosis and long tailed distribution is extended to the NLAR(p) model. A necessary and sufficient condition for the existence of an innovation sequence $\{e_n\}$ and a stationary ergodic NLAR(p) model $\{X_n\}$ is derived. It is shown that the distribution of the innovation sequence $\{e_n\}$ is given by a probabilistic mixture of independent Laplace distributions if $0 < \alpha_0 < 1$ and a probabilistic mixture of independent Laplace distributions and a degenrate distribution if $\alpha_0 = 0$.

Estimation of parameters α_i and β_i, $i = 1, 2, \ldots, p$ is of importance to predict the NLAR(p) model and will be treated elsewhere.

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