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The Bahadur Efficiency of the Power-Divergence Statistics Conditional on Margins for Testing Homogeneity with Equal Sample Size

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Abstract

The family of power-divergence statistics conditional on margins is considered for testing homogeneity of r multinomial populations with equal sample size and the exact Bahadur slope is obtained. It is shown that the likelihood ratio test conditional on margins is the most Bahadur efficient among the family of power-divergence statistics.

Key Words : Product multinomial; Nuisance parameter; Conditional test; Likelihood ratio test; Limiting conditional distribution.

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1. INTRODUCTION

Let $\mathbf{X}_1, \ldots, \mathbf{X}_r$ be a sequence of independent r random vectors with multinomial distribution

$$P[\mathbf{X}_i = \mathbf{x}_i] = n! \prod_{j=1}^{c} \frac{p_{ij}^{x_{ij}}}{x_{ij}!}.$$

We can make an $r \times c$ contingency table whose row margins are fixed. We are interested in testing homogeneity of r multinomial populations.

$$H_0 : p_{ij} = p_j, \quad i = 1, \ldots, r, \quad j = 1, \ldots, c.$$

Cressie and Read (1984) proposed the family of power-divergence statistics $2nI^{\lambda}(\mathbf{X}^{r\times c}/n, \{\hat{p}_j\})$ for goodness-of-fit test.

$$2nI^{\lambda}(\mathbf{X}^{r\times c}/n, \{\hat{p}_j\}) = \frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{r} \sum_{j=1}^{c} X_{ij} \left[\left(\frac{X_{ij}}{X_{+j}/r} \right)^{\lambda} - 1 \right], \quad -\infty < \lambda < \infty$$

where for $\lambda = -1, 0$, the limit as $\lambda \to -1, 0$ is used. $2nI^{\lambda}(\mathbf{X}^{r\times c}/n, \{\hat{p}_j\})$ covers several test statistics

$$\begin{aligned}
\lambda = 0 &\longrightarrow \text{likelihood ratio statistic}\\
\lambda = 1 &\longrightarrow \text{chi-square statistic}\\
\lambda = -1 &\longrightarrow \text{modified likelihood ratio statistic}\\
\lambda = -1/2 &\longrightarrow \text{Freeman-Tukey statistic}\\
\lambda = -2 &\longrightarrow \text{Neyman-modified chi-square statistic.}
\end{aligned}$$

These test statistics have a limiting chi-square distribution with $(r-1)(c-1)$ degrees of freedom under the null hypothesis (Read and Cressie 1988, p50). The question arises the accuracy of the chi-square approximation over the parameter space, because the exact distribution of $2nI^{\lambda}(\mathbf{X}^{r\times c}/n, \{\hat{p}_j\})$ under the null hypothesis depends on nuisance parameters. Loh (1989,1993) showed that the limiting sizes (the supremum of the probability of type I error over the parameter space) of the chi-square test ($\lambda = 1$) and the likelihood ratio test ($\lambda = 0$) are always greater than the nominal level. Since conditioning is a standard way of eliminating nuisance parameters in contingency tables, $2nI^{\lambda}(\mathbf{X}^{r\times c}/n, \{\hat{p}_j\})$ *conditionally* given the column margins (the sufficient statistics for the nuisance parameters under the null hypothesis) is suggested to remove dependency on nuisance parameters. From now, conditional power-divergence statistics means power-divergence statistics conditional on column margins.

Kang (1997) developed the Bahadur efficiency for conditional tests and obtain the Bahadur efficiency of the conditional likelihood ratio test relative to the conditional chi-square test of independence in two-way contingency tables. Cressie and Read (1984) showed that, in *unconditional* case, no member of the power-divergence family can be more Bahadur efficient than the likelihood ratio test ($\lambda = 0$) when hypothesized model does not require parameter estimation.

The aim of this study is to obtain exact Bahadur slope of the conditional power-divergence statistics and show that the conditional likelihood ratio test is the most Bahadur efficient among the family of the conditional power-divergence statistics.

2. BAHADUR EFFICIENCY FOR CONDITIONAL TESTS

In this section we develop Bahadur efficiency for conditional tests like unconditional cases. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent, identically distributed random variables whose common probability distribution is P_θ where $\theta = (\eta, \psi) \in \Theta$, η is a p-dimensional vector in Δ, ψ is a q-dimensional vector in Ψ. Let Δ_0 be a proper subset of Δ. Let $\Theta_0 = \{\theta = (\eta, \psi) : \eta \in \Delta_0, \psi \in \Psi\}$ and $\Theta_1 = \Theta - \Theta_0$.

For testing $H_0 : \theta \in \Theta_0$ with a nuisance parameter ψ, let S_n and T_n be two statistics. We are interested in assessing the performance of the conditional statistic $S_n|T_n$ for testing H_0 as $n \to \infty$. Assume that large values of S_n are significant.

Let conditional sizes and conditional type II error probabilities be

$$\alpha_n(T_n) = \sup_{\theta \in \Theta_0} P_\theta(S_n > s_n|T_n) \quad \text{and} \quad \beta_n(\theta|T_n) = P_\theta(S_n \le s_n|T_n).$$

Definition. For $\theta_1 \in \Theta_1$, let the critical value $s_n = s_n(\theta_1)$ so that

$$\beta_n(\theta_1|T_n) = P_{\theta_1}[S_n \le s_n(\theta_1)|T_n] \xrightarrow{P_{\theta_1}} \beta(\theta_1), \quad 0 < \beta(\theta_1) < 1.$$

The sequence of the conditional test statistics $S_n|T_n$ has exact Bahadur slope $c(\theta_1)$ if

$$-\frac{1}{n}\ln(\alpha_n(\theta_1|T_n)) \xrightarrow{P_{\theta_1}} \frac{1}{2}c(\theta_1)$$

as $n \to \infty$ for $\theta_1 \in \Theta_1$, where

$$\alpha_n(\theta_1|T_n) = \sup_{\theta_0 \in \Theta_0} P_{\theta_0}[S_n \ge s_n(\theta_1)|T_n]$$

and $\xrightarrow{P_{\theta_1}}$ means convergence in probability.

Definition. For two sequences of conditional test statistics $\{S_n^{(i)}|T_n^{(i)}\}, i = 1, 2$, we define

$$e_{12}(\theta_1) = \frac{c_1(\theta_1)}{c_2(\theta_1)}$$

as the Bahadur efficiency of $\{S_n^{(1)}|T_n^{(1)}\}$ relative to $\{S_n^{(2)}|T_n^{(2)}\}$ at θ_1 for every $\theta_1 \in \Theta_1$.

3. PRELIMINARY RESULTS

We introduce a result on the limiting form of the conditional distribution. Suppose that a random vector $(\mathbf{U}_n, \mathbf{V}_n)$ converges in distribution to a random vector $(\mathbf{U}, \mathbf{V})$. Steck (1957, p247-248 and p254) gives a sufficient condition that the conditional distribution of $\mathbf{U}_n$, given $\mathbf{V}_n = \mathbf{v}_n$, converges in distribution to the conditional distribution of $\mathbf{U}$, given $\mathbf{V} = \mathbf{v}$ as $\mathbf{v}_n \longrightarrow \mathbf{v}$. We use his result to obtain the limiting conditional distribution. We restate Steck's result using our notation.

Suppose, for $k = 1, \ldots, n$, that we observe a sequence of independent $r \times c$ random matrices

$$\begin{pmatrix} X^{(n)}[1,1,k] & \ldots & X^{(n)}[1,c,k] \\ \vdots & & \vdots \\ X^{(n)}[r,1,k] & \ldots & X^{(n)}[r,c,k] \end{pmatrix}.$$

Let

$$X^{(n)}[+,j,k] = \sum_{i=1}^{r} X^{(n)}[i,j,k], \qquad j = 1, \ldots, c-1.$$

and define, for $k = 1, \ldots, n$, the column vectors

$$\mathbf{X}^{(n)}_{+,(c-1),k} = (X^{(n)}[+,1,k], \ldots, X^{(n)}[+,c-1,k])^t.$$

Without loss of generality we assume all the random variables have zero means. We introduce notation to use Steck's result. Let

$$\begin{aligned} A^{(n)}[i,j,+] &= \{\sum_{k=1}^{n} E(X^{(n)}[i,j,k])^2\}^{1/2}, \\ Y^{(n)}[i,j,+] &= \sum_{k=1}^{n} X^{(n)}[i,j,k]/A^{(n)}[i,j,+], \end{aligned}$$

$$\begin{aligned} A^{(n)}[+,j,+] &= \{\sum_{k=1}^{n} E(X^{(n)}[+,j,k])^2\}^{1/2}, \\ Y^{(n)}[+,j,+] &= \sum_{k=1}^{n} X^{(n)}[+,j,k]/A^{(n)}[+,j,+], \end{aligned}$$

and

$$\begin{aligned} \mathbf{Y}^{(n)}_{(r,c),+} &= (Y^{(n)}[1,1,+],\ldots,Y^{(n)}[1,c,+],Y^{(n)}[2,1,+],\ldots, \\ &\quad Y^{(n)}[2,c,+],\ldots,Y^{(n)}[r,1,+],\ldots,Y^{(n)}[r,c,+])^t \\ \mathbf{Y}^{(n)}_{+,(c-1),+} &= (Y^{(n)}[+,1,+],\ldots,Y^{(n)}[+,c-1,+])^t \end{aligned}$$

$$\begin{aligned} \mathbf{Z}^{(n)}_k &= \left(\frac{X^{(n)}[1,1,k]}{A^{(n)}[1,1,+]},\ldots,\frac{X^{(n)}[1,c,k]}{A^{(n)}[1,c,+]},\frac{X^{(n)}[2,1,k]}{A^{(n)}[2,1,+]},\ldots,\frac{X^{(n)}[2,c,k]}{A^{(n)}[2,c,+]},\right. \\ &\quad \left.\ldots,\frac{X^{(n)}[r,1,k]}{A^{(n)}[r,1,+]},\ldots,\frac{X^{(n)}[r,c,k]}{A^{(n)}[r,c,+]},\frac{X^{(n)}[+,1,k]}{A^{(n)}[+,1,+]},\ldots,\frac{X^{(n)}[+,c-1,k]}{A^{(n)}[+,c-1,+]}\right)^t \\ \mathbf{Z}^{(n)} &= \left((\mathbf{Y}^{(n)}_{(r,c),+})^t,(\mathbf{Y}^{(n)}_{+,(c-1),+})^t\right)^t. \end{aligned}$$

We denote the covariance matrix of $\mathbf{Z}^{(n)}$ by Q_n. Let $Q_{n11} = Var(\mathbf{Y}^{(n)}_{(r,c),+})$, $Q_{n12} = Cov(\mathbf{Y}^{(n)}_{(r,c),+},\mathbf{Y}^{(n)}_{+,(c-1),+})$, and $Q_{n22} = var(\mathbf{Y}^{(n)}_{+,(c-1),+})$, then

$$Q_n = \begin{pmatrix} Q_{n11} & Q_{n12} \\ Q^t_{n12} & Q_{n22} \end{pmatrix} \tag{3.1}$$

If Q_n tends to a fixed nonnegative definite matrix Q, we will let Q_{11}, Q_{12}, and Q_{22} be corresponding limits of Q_{n11}, Q_{n12}, and Q_{n22} respectively. Using the above notation we have the following theorem.

Theorem 1. (Steck 1957, p244) Assume that

1. $\mathbf{X}^{(n)}_{+,(c-1),k}$ is distributed on a lattice with positive maximum span $\mathbf{h} = (h_{+1}\mathbf{e}_1,\ldots,h_{+,(c-1)}\mathbf{e}_{c-1})$ which is independent of k and n, where h_{+j} is the maximum span of $X^{(n)}[+,j,k]$ and $\mathbf{e}_i = (0,\ldots,\underbrace{1}_{i-th\ coordinate},\ldots,0)$.
2. $$\lim_{n\to\infty} \inf\{A^{(n)}[i,j,+] : i = 1,\ldots,r, j = 1,\ldots,c\} = \infty.$$

$$\lim_{n\to\infty} \inf\{A^{(n)}[+,j,+] : j = 1,\ldots,c-1\} = \infty.$$

3.

$$\lim_{n\to\infty}\left(\prod_{j=1}^{c-1} A^{(n)}[+,j,+]\right)\exp[-\sum_{k=1}^{n} a(k)] = 0,$$

where $a(k)$ is the indicated positive constant for the characteristic function of $\mathbf{Z}_k^{(n)}$ in the sense of Lemma 2.4 in Steck (1957).

4. Q_n converges to a fixed nonnegative definite matrix Q for which Q_{22} is positive definite.

5. For some $\delta > 0, \lim_{n\to\infty}\sum_{k=1}^{n} E|\mathbf{Z}_k^{(n)}|^{2+\delta} = \mathbf{0}$ and

$$\limsup_{n}\sup_{j}\sup_{k} nE|X^{(n)}[+,j,k]|^{2+\delta}/(A^{(n)}[+,j,+])^2 \le C_1 < \infty.$$

Then

$$\lim_{n\to\infty}|w_n(\alpha,\mathbf{v}) - w(\alpha,\mathbf{v})| = 0,$$

uniformly on bounded subsets of A_n, where

- $w_n(\alpha,\mathbf{v})$ is the conditional characteristic function of $\mathbf{Y}^{(n)}_{(r,c),+}$, given $\mathbf{Y}^{(n)}_{+,(c-1),+} = \mathbf{v}$,
- A_n is the set of possible values of $\mathbf{Y}^{(n)}_{+,(c-1),+}$,
- $w(\alpha,\mathbf{v}) = \exp\{i\alpha^t Q_{12}Q_{22}^{-1}\mathbf{v} - \frac{1}{2}\alpha^t Q_{11.2}\alpha\}$,
- $Q_{11.2} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^t$.

The conditional probability distribution of $\mathbf{X}^{r\times c}$ under the alternative, given the margins $X_{+j} = \sum_{i=1}^{r} X_{ij}, j = 1,\ldots,c-1$, is a noncentral multivariate hypergeometric distribution (Cornfield 1956). We will prove asymptotic normality of the distribution using Steck's result.

Theorem 2. Let

$$Z_{ij}^{(n)} = \frac{\sqrt{n}}{\sqrt{p_{ij}(1-p_{ij})}}\left(\frac{X_{ij}}{n} - p_{ij}\right) \quad \text{for} \quad i = 1,\ldots,r, \quad j = 1,\ldots,c,$$

$$Z_{+j}^{(n)} = \frac{\sqrt{n}}{\sqrt{\sum_{i=1}^{r} p_{ij}(1-p_{ij})}}\left(\frac{X_{+j}}{n} - \sum_{i=1}^{r} p_{ij}\right) \quad \text{for} \quad j = 1,\ldots,c-1,$$

and let

$$\mathbf{Z}^{(n)}_{(r,c)} = (Z^{(n)}_{11}, Z^{(n)}_{12}, \ldots, Z^{(n)}_{1c}, Z^{(n)}_{21}, \ldots, Z^{(n)}_{2c}, \ldots, Z^{(n)}_{r1}, \ldots, Z^{(n)}_{rc})^t$$

$$\mathbf{Z}^{(n)}_{+\bullet} = (Z^{(n)}_{+1}, \ldots, Z^{(n)}_{+,c-1})^t$$

$$R_n = \{\mathbf{T}^{(n)} : -C \leq Z^{(n)}_{+j} \leq C, j = 1, \ldots, c-1\}$$

where

$$\mathbf{T}^{(n)} = \left(\frac{X_{+1}}{n}, \ldots, \frac{X_{+,c-1}}{n}\right)^t$$

and C is a sufficiently large real number.

Then, the conditional distribution of $\mathbf{Z}^{(n)}_{(r,c)}$, given $\mathbf{T}^{(n)} = \mathbf{t}^{(n)}, \mathbf{t}^{(n)} \in R_n$, converges in distribution as n goes to ∞ to the rc dimensional multivariate singular normal distribution with mean vector μ and covariance matrix $\Sigma_{11.2}$, where μ and $\Sigma_{11.2}$ can be obtained from Theorem 1.

Proof. For $\mathbf{t}_n \in R_n$, the condition $\mathbf{T}_n = \mathbf{t}_n$ implies that each element of $\mathbf{Z}^{(n)}_{\bullet+}$ is bounded by definition of R_n. We check the conditions of Theorem 1 to obtain the limiting distribution of the conditional distribution of $\mathbf{Z}^{(n)}_{(r,c)}$, given that each element of $\mathbf{Z}^{(n)}_{\bullet+}$ is bounded.

Define $X^{(n)}[i,j,k]$, for $i = 1, \ldots, r, j = 1, \ldots, c$ to be

$$X^{(n)}[i,j,k] = \begin{cases} 1 - p_{ij} & \text{if an observation of } i\text{th multinomial trial in the} \\ & k\text{th trial belongs to the } j\text{th column category} \\ -p_{ij} & \text{otherwise.} \end{cases}$$

Note that $X^{(n)}[i,j,k]$ does not depend on n and k. From the definition of $X^{(n)}[i,j,k]$, we can further define $X^{(n)}[+,j,k]$, $\mathbf{X}^{(n)}_{+,(r-1),k}$.

Then

$$A^{(n)}[i,j,+] = \{\sum_{k=1}^{n} E[X^{(n)}[i,j,k]]^2\}^{1/2} = [np_{ij}(1-p_{ij})]^{1/2}$$

$$A^{(n)}[+,j,+] = [n \sum_{i=1}^{r} p_{ij}(1-p_{ij})]^{1/2} \quad \text{for} \quad j = 1, \ldots, c-1$$

$$Y^{(n)}[i,j,+] = \sum_{k=1}^{n} X^{(n)}[i,j,k]/A^{(n)}[i,j,+] = \frac{\sqrt{n}}{\sqrt{p_{ij}(1-p_{ij})}}\left(\frac{X_{ij}}{n} - p_{ij}\right) = Z^{(n)}_{ij}$$

$$Y^{(n)}[+,j,+] = Z_{+j}, \text{ for } j = 1, \ldots, c-1.$$

Since two sets of notation match, the conditional distribution of $\mathbf{Y}^{(n)}_{(r,c)}$ given $\mathbf{Y}^{(n)}_{+,(c-1),+} = \mathbf{v}$ is the same as the conditional distribution of $\mathbf{Z}^{(n)}_{(r,c)}$ given $\mathbf{Z}^{(n)}_{+\bullet} = \mathbf{v}$.

For condition 1 of Theorem 1, since $X^{(n)}[+,j,k]$ do not depend on k and n, neither do $\mathbf{X}^{(n)}_{+.(c-1).k}$. Hence, $\mathbf{h}$ is independent of k and n. Condition 2 is trivially satisfied.

For condition 3, note that positive constant $a(k)$ is the indicated constant for the characteristic function of $\mathbf{Z}^{(n)}_k$. Since $\mathbf{Z}^{(n)}_k$ does not depend on k, $a(k) = a$ independent of k. Thus,

$$\lim_{n\to\infty}\left(\prod_{j=1}^{c-1}\sqrt{n\sum_{i=1}^{r}p_{ij}(1-p_{ij})}\right)\exp[-\sum_{k=1}^{n}a(k)] = 0.$$

For condition 4, Q_n does not depend on n and so is a limiting moment matrix Q. Since $\mathbf{b}'\mathbf{Z}^{(n)}_{+\bullet}$ does not have a degenerate distribution for any $\mathbf{b}' \neq \mathbf{0}$, Q_{22} is positive definite.

For condition 5, take $\delta = 1$. Since by straightforward calculation,

$$E\left(\frac{X^{(n)}[i,j,k]}{A^{(n)}[i,j,+]}\right)^3 = \frac{1-2p_{ij}}{n^{3/2}\sqrt{p_{ij}(1-p_{ij})}},$$

$$E\left(\frac{X^{(n)}[+,j,k]}{A^{(n)}[+,j,+]}\right)^3 = \frac{\sum_{i=1}^{r}(1-p_{ij})(1-2p_{ij})p_{ij}}{(n\sum_{i=1}^{r}p_{ij}(1-p_{ij}))^{3/2}}$$

does not depend on k, $\lim_{n\to\infty}\sum_{k=1}^{n}E|\mathbf{Z}^{(n)}_k|^3 = \mathbf{0}$. Since

$$\frac{nE|X^{(n)}[+,j,k]|^3}{(A^{(n)}[+,j,+])^2} = \frac{\sum_{i=1}^{r}(1-p_{ij})(1-2p_{ij})p_{ij}}{\sum_{i=1}^{r}(1-p_{ij})p_{ij}}$$

does not depend on k and n, condition 5 is satisfied.

4. MAIN RESULT

First, we let the conditional type II error probability go to β, $0 < \beta < 1$, as $n \to \infty$, under the alternative hypothesis.

Lemma 1. Under the alternative hypothesis, for $\mathbf{t}_n \in R_n$, if we take critical value

$$s_{n,\mathbf{t}_n,\mathbf{p}} = z_\beta\sqrt{n}\sqrt{\mathbf{w}_n^t\Sigma_{11.2}\mathbf{w}_n} + \sqrt{n}\mathbf{w}_n^t\mu + nJ,$$

then

$$\beta_n(\mathbf{p}|\mathbf{t}_n) \longrightarrow \beta, \quad 0 < \beta < 1,$$

where

$$J = \frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{r} \sum_{j=1}^{c} p_{ij} \left[\left(\frac{p_{ij}}{\bar{p_j}} \right)^{\lambda} - 1 \right] + o(1), \quad \bar{p_j} = \frac{1}{r} \sum_{i=1}^{r} p_{ij}$$

and $\mathbf{w}_n$ is rc dimensional column vector such that

$$\frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{\sqrt{p_{ij}(1-p_{ij})}[p_{ij}^{\lambda} - (\bar{p_j} + o(1))^{\lambda} + p_{ij}^{\lambda}\lambda]}{(\bar{p_j} + o(1))^{\lambda}} Z_{ij}^{(n)} = \mathbf{w}_n^t \mathbf{Z}_{(r,c)}^{(n)}$$

and z_β is $100\beta\%$ percentile of the standard normal distribution with $0 < \beta < 1$ and μ and $\Sigma_{11.2}$ are from Theorem 1.

Proof.

$$\begin{aligned} & \frac{1}{n} 2I^{\lambda}(\mathbf{X}^{r\times c}, \{\hat{p}_j\}), \quad \text{given} \quad \mathbf{t}_n \in R_n \\ = \; & J + \frac{1}{\sqrt{n}} \mathbf{w}_n^t \mathbf{Z}_{(r,c)}^{(n)} + o_p\left(\frac{1}{\sqrt{n}}\right) \end{aligned}$$

from $x^{\lambda} = 1 + \lambda x + o(x)$ and $X_{ij} = np_{ij} + \sqrt{n}\sqrt{p_{ij}(1-p_{ij})} Z_{ij}^{(n)}$
$X_{+j}/(nr) = \bar{p_j} + o(1)$.

From theorem 2, the result is obtained.

Let

$$\begin{aligned} D_n^* \; = \; & \left\{ \frac{1}{n}\mathbf{x} \middle| \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{x_{ij}}{n} \left[\left(\frac{x_{ij}}{x_{+j}/r} \right)^{\lambda} - 1 \right] \geq \sum_{i=1}^{r} \sum_{j=1}^{c} p_{ij} \left[\left(\frac{p_{ij}}{\bar{p_j}} \right)^{\lambda} - 1 \right], \right. \\ & \left. \left| \frac{1}{n} \sum_{i=1}^{r} x_{ij} - \sum_{i=1}^{r} p_{+j} \right| < \frac{C\sqrt{\sum_{i=1}^{r} p_{ij}(1-p_{ij})}}{\sqrt{n}}, j = 1, \ldots, c \right\} \end{aligned}$$

$$D = \left\{ \mathbf{p}^0 \;\middle|\; \sum_{i=1}^{r} \sum_{j=1}^{c} p_{ij}^0 \left[\left(\frac{p_{ij}^0}{\bar{p_j}} \right)^{\lambda} - 1 \right] \geq \sum_{i=1}^{r} \sum_{j=1}^{c} p_{ij} \left[\left(\frac{p_{ij}}{\bar{p_j}} \right)^{\lambda} - 1 \right], \quad \sum_{j=1}^{c} p_{ij}^0 = 1, 0 < p_{ij}^0 < 1 \right\}$$

$$\begin{aligned} D_n(\mathbf{t}_n) \; = \; & \left\{ \frac{1}{n}\mathbf{x} \middle| \sum_{i=1}^{r} \sum_{j=1}^{c} \frac{x_{ij}}{n} \left[\left(\frac{x_{ij}}{x_{+j}/r} \right)^{\lambda} - 1 \right] \geq \sum_{i=1}^{r} \sum_{j=1}^{c} p_{ij} \left[\left(\frac{p_{ij}}{\bar{p_j}} \right)^{\lambda} - 1 \right] + o(1), \right. \\ & \left. \sum_{j=1}^{c} x_{ij} = n, i = 1, \ldots, r, \sum_{i=1}^{r} x_{ij} = x_{+j}^{\star}, j = 1, \ldots, c \right\}, \end{aligned}$$

where $o(1)$ depends on $\mathbf{t}_n$ and $\mathbf{x}$ is an $r \times c$ table whose element x_{ij} is an integer $0 \le x_{ij} \le n$, $(x^*_{+1}, \ldots, x^*_{+c})$ are given column margins for $\mathbf{t}_n \in R_n$.

Lemma 2.

$$\lim_{n\to\infty} \inf \left\{ \sum_{i=1}^{r} \sum_{j=1}^{c} p^0_{ij(n)} \ln \left(\frac{p^0_{ij(n)}}{\bar{p_j}} \right) : \mathbf{p}^0_n \in D^*_n \right\}$$
$$= \inf \left\{ \sum_{i=1}^{r} \sum_{j=1}^{c} p^0_{ij} \ln \left(\frac{p^0_{ij}}{\bar{p_j}} \right) : \mathbf{p}^0 \in D \right\}$$

The proof can be done using the similar argument of Bahadur's result (1971, p17-18).

Lemma 3. For given $\mathbf{t}_n \in R_n$,

$$\lim_{n\to\infty} \inf \left\{ g_n(\mathbf{x}, \mathbf{t}_n) : \frac{1}{n}\mathbf{x} \in D_n(\mathbf{t}_n) \right\}$$
$$= \lim_{n\to\infty} \inf \left\{ \sum_{i=1}^{r} \sum_{j=1}^{c} p^0_{ij(n)} \ln \left(\frac{p^0_{ij(n)}}{\bar{p_j}} \right) : \mathbf{p}^0_n \in D^*_n \right\}$$

where

$$g_n(\mathbf{x}, \mathbf{t}_n) = \sum_{i=1}^{r} \sum_{j=1}^{c} \left(\frac{x_{ij}}{n} \right) \ln \left(\frac{x_{ij}}{n} \right) - \sum_{j=1}^{c} r\bar{p_j} \ln(\bar{p_j}) + o(1). \tag{4.2}$$

For proof, see Kang (1997, Lemma 4.3).

Theorem 3. The exact Bahadur slope of the conditional power-divergence statistic for the test of homogeneity of r multinomial populations is

$$2 \inf \left\{ \sum_{i=1}^{r} \sum_{j=1}^{c} p^0_{ij} \ln \left(\frac{p^0_{ij}}{\bar{p_j}} \right) : \mathbf{p}^0 \in D \right\}$$

where

$$\bar{p_j} = \frac{1}{r} \sum_{i=1}^{r} p_{ij}.$$

Proof. Lemma 1 implies that the conditional type II error probability goes to β under the alternative hypothesis, if the rejection region is

$$D_n(\mathbf{t}_n) = \left\{ \frac{1}{n}\mathbf{x} \,\middle|\, 2I^\lambda(\mathbf{x}^{r\times c}, \{\hat{p}_j\}) \ge s_{n,\mathbf{t}_n,\mathbf{p}} \right\}$$

$$= \left\{ \frac{1}{n}\mathbf{x} \middle| \sum_{i=1}^{r}\sum_{j=1}^{c} \frac{x_{ij}}{n}\left[\left(\frac{x_{ij}}{x_{+j}/r}\right)^{\lambda} - 1\right] \geq \sum_{i=1}^{r}\sum_{j=1}^{c} p_{ij}\left[\left(\frac{p_{ij}}{\bar{p_j}}\right)^{\lambda} - 1\right] + o(1), \right.$$
$$\left. \sum_{j=1}^{c} x_{ij} = n, i = 1, \ldots, r, \sum_{i=1}^{r} x_{ij} = x^\star_{+j}, j = 1, \ldots, c \right\},$$

where $o(1)$ depends on $\mathbf{t}_n$ and $\mathbf{x}$ is an $r \times c$ table whose element x_{ij} is an integer $0 \leq x_{ij} \leq n$, $(x^\star_{+1}, \ldots, x^\star_{+c})$ are given column margins for $\mathbf{t}_n \in R_n$.

We investigate the rate at which the conditional size converges to zero. For $\mathbf{t}_n \in R_n$,

$$\alpha(\mathbf{p}|\mathbf{t}_n) = \sum_{\frac{1}{n}\mathbf{x} \in D_n(\mathbf{t}_n)} \left[\frac{(n!)^r \prod_{j=1}^{c}(x_{+j}!)}{(nr)! \prod_{i=1}^{r}\prod_{j=1}^{c} x_{ij}!}\right].$$

Since

$$\sup\left\{ \frac{(n!)^r \prod_{j=1}^{c}(x_{+j}!)}{(nr)! \prod_{i=1}^{r}\prod_{j=1}^{c} x_{ij}!} \middle| \frac{1}{n}\mathbf{x} \in D_n(\mathbf{t}_n) \right\} \leq \alpha(\mathbf{p}|\mathbf{t}_n)$$
$$\leq \tau \sup\left\{ \frac{(n!)^r \prod_{j=1}^{c}(x_{+j}!)}{(nr)! \prod_{i=1}^{r}\prod_{j=1}^{c} x_{ij}!} \middle| \frac{1}{n}\mathbf{x} \in D_n(\mathbf{t}_n) \right\},$$

where τ is the cardinality of set $D_n(\mathbf{t}_n)$,

$$\begin{aligned} &- \lim_{n\to\infty} \frac{1}{n} \ln(\alpha(\mathbf{p}|\mathbf{t}_n)) \\ = \ &\lim_{n\to\infty} \inf\left\{ -\frac{1}{n} \ln\left[\frac{(n!)^r \prod_{j=1}^{c}(x_{+j}!)}{(nr)! \prod_{i=1}^{r}\prod_{j=1}^{c} x_{ij}!}\right]\right\} \end{aligned}$$

provided we prove $\ln(\tau)/n \to 0$ as $n \to \infty$.

From Stirling's formula $\ln(m!) = m\ln(m) - m + o(m)$, and $\mathbf{t}_n \in R_n$ implies $x_{+j}/n = r\bar{p_j} + o(1)$,

$$\begin{aligned} &-\frac{1}{n}\ln\left[\frac{(n!)^r \prod_{j=1}^{c}(x_{+j}!)}{(nr)! \prod_{i=1}^{r}\prod_{j=1}^{c} x_{ij}!}\right] \\ = \ &\sum_{i=1}^{r}\sum_{j=1}^{c} \frac{x_{ij}}{n}\ln\left(\frac{x_{ij}}{n}\right) - \sum_{j=1}^{c} r\bar{p_j}\ln(\bar{p_j}) + o(1) \\ \equiv \ &g_n(\mathbf{x}, \mathbf{t}_n) \end{aligned}$$

Therefore, for $\mathbf{t}_n \in R_n$, by Lemma2 and Lemma3,

$$- \lim_{n\to\infty} \frac{1}{n}\ln(\alpha(\mathbf{p}_1|\mathbf{t}_n)) = \lim_{n\to\infty} \inf\left\{ g_n(\mathbf{x}, \mathbf{t}_n) : \frac{1}{n}\mathbf{x} \in D_n(\mathbf{t}_n)\right\}$$

$$= \lim_{n\to\infty} \inf \left\{ \sum_{i=1}^{r}\sum_{j=1}^{c} p^0_{ij(n)} \ln\left(\frac{p^0_{ij(n)}}{\bar{p}_j}\right) : \mathbf{p}^0_n \in D^*_n \right\}$$

$$= \inf \left\{ \sum_{i=1}^{r}\sum_{j=1}^{c} p^0_{ij} \ln\left(\frac{p^0_{ij}}{\bar{p}_j}\right) : \mathbf{p}^0 \in D \right\}$$

Since $P(R_n) \longrightarrow 1$, as $n \longrightarrow \infty$,

$$-\lim_{n\to\infty} \frac{1}{n} \ln(\alpha(\mathbf{p}|\mathbf{T}_n)) \xrightarrow{P} \inf \left\{ \sum_{i=1}^{r}\sum_{j=1}^{c} p^0_{ij} \ln\left(\frac{p^0_{ij}}{\bar{p}_j}\right) : \mathbf{p}^0 \in D \right\}.$$

We complete proof by showing $\lim_{n\to\infty} \frac{1}{n}\ln(\tau) = 0$. $\tau \leq \tau^*$, where τ^* is cardinality of

$$\left\{ \frac{1}{n}\mathbf{x} \,\middle|\, \sum_{j=1}^{c} x_{ij} = n, i = 1, \ldots, c \right\}.$$

From Feller (1957, p39), we can show $\lim_{n\to\infty} \frac{1}{n}\ln(\tau^*) \longrightarrow 0$ easily by using Stirling's formula.

The right hand side of nonlinear constraint in D is always nonnegative (Read and Cressie 1988, p110) and vanishes when $\lambda = 0$. Therefore, the exact slope has the maximum at $\lambda = 0$ (the conditional likelihood ratio test).

The conditional likelihood ratio test is more Bahadur efficient than $2nI^\lambda$ $(\mathbf{X}^{r\times c}/n, \{\hat{p}_j\})$ which Read and Cressie (1988) advocate.

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