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Test for Parameter Changes in the AR(1) Process†

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Abstract

In this paper the parameter change problem in the stationary time series is considered. We propose a cumulative sum (CUSUM) of squares-type test statistic for detection of parameter changes in the AR(1) process. The proposed test statistic is based on the CUSUM of the squared observations and is shown to converge to a standard Brownian bridge. Simulations are performed to evaluate the performance of the proposed statistic and a real example is provided to illustrate the procedure.

Key Words : Cusum; Change points; Brownian bridge.

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1. INTRODUCTION

Let $\{x_t\}$ follow an AR(1) process

$$x_t = \phi x_{t-1} + u_t, \quad t = 1, \ldots, T, \tag{1.1}$$

where $|\phi| < 1$ and u_t is i.i.d. $N(0, \sigma_u^2)$. Usually, the parameter ϕ is assumed to be constant over time. However, it has been pointed out by many authors that the assumption is often inadequate.

Nicholls and Quinn (1982) considered the random coefficient AR model. Tyssedal and Tjøstheim (1988) suggested AR models where the parameters are piecewise constant and change according to a Markov chain mechanism. Another approaches can be found in Tjøstheim (1986) and McCulloch and Tsay (1993). However, all of these are focused on modeling under the assumption that the parameter is unstable. On the other hand, many authors have considered tests for changes of the AR parameter ϕ and noise variance σ_u^2. Wichern *et al.* (1976), Davis (1979), and Ryu and Cho (1987) studied the variance change problem with constant AR parameter. Kwoun (1988), Bai (1993), and Davis *et al.* (1995) studied the AR parameter change problem with constant noise variance.

In this paper we propose a test statistic for parameter changes in the AR(1) process. Note that under mild conditions, the AR(1) process is weakly stationary and the variance function is given by $Var(x_t) = \sigma_u^2/(1-\phi^2)$. Since the variance is a function of the parameters which constitute the process changes in the parameter result in variance changes. Therefore, to detect parameter changes, one can also use the test procedure based upon the CUSUM of squares by Inclán and Tiao (1994) which was used to detect changes in the variance.

In Section 2, we propose a test statistic and the detection procedure for parameter changes in the AR(1) process. Simulation results are reported by comparing the performance of the proposed test with the other test in Section 3. Finally, we apply the proposed test statistic to a real example.

2. CUSUM OF SQUARES-TYPE TEST STATISTIC

It is shown that the AR(1) process is a mixingale in the sense of McLeish (1975) in the following lemma.

Lemma 1. Assume that $\{x_t\}$ follows the AR(1) process in (1.1) with $\sigma_x^2 = E(x_t^2) = \sigma_u^2/(1-\phi^2)$. Define $\mathcal{F}^t = \sigma(u_{t-1}, u_{t-2}, \ldots)$. Then $\{(x_t^2 - \sigma_x^2, \mathcal{F}^t)\}$ is a mixingale of order $-1/2$.

Proof. By recursion, we obtain

$$\begin{aligned} x_t &= \phi^m x_{t-m} + \sum_{j=0}^{m-1} \phi^j u_{t-j}, \\ x_t^2 &= \phi^{2m} x_{t-m}^2 + 2\phi^m x_{t-m} \sum_{j=0}^{m-1} \phi^j u_{t-j} + (\sum_{j=0}^{m-1} \phi^j u_{t-j})^2. \end{aligned}$$

Thus,

$$\begin{aligned} \|E(x_t^2 - \sigma_x^2)|\mathcal{F}^{t-m}\|_2 &= \|\phi^{2m} x_{t-m}^2 + \sigma_u^2 \sum_{j=0}^{m-1} \phi^{2j} - \sigma_x^2\|_2 \\ &= \|\phi^{2m} x_{t-m}^2 - \phi^{2m}\sigma_u^2/(1-\phi^2)\|_2 \\ &\leq \phi^{2m}[\|x_{t-m}^2\|_2 + \sigma_u^2/(1-\phi^2)] \\ &\stackrel{\text{def}}{=} \psi_m c_t. \end{aligned}$$

The last equation is obtained from the definition of a mixingale in MeLeish (1975) when there exists a positive sequence $\{\psi_m\}$ such that $\psi_m \to 0$ as $m \to \infty$ and a nonnegative sequence $\{c_t\}$ for $t \geq 0$ and $k \geq 0$.

Lemma 2. Under the same conditions as in Lemma 1,

$$E\{(S_{k+T} - S_k)^2|\mathcal{F}^{k-m}\}/T \to \sigma^2$$

in L_1 norm as $min(m, k, T) \to \infty$, where $\sigma^2 = E\xi_0^2 + 2\sum_{k=1}^{\infty} E\xi_0\xi_k = 2\sigma_u^4(1+\phi^2)/(1-\phi^2)^3$.

Proof. Define $\Delta = E[(S_{k+T} - S_k)^2|\mathcal{F}^{k-m}]/T$, then

$$\begin{aligned} \Delta &= \frac{1}{T}\sum_{t=1}^{T} E[(x_{t+k}^2 - \sigma_x^2)^2|\mathcal{F}^{k-m}] \\ &\quad + \frac{2}{T}\sum_{t=1}^{T-1}\sum_{s=t+1}^{T} E[(x_{t+k}^2 - \sigma_x^2)(x_{s+k}^2 - \sigma_x^2)|\mathcal{F}^{k-m}] \\ &\stackrel{\text{def}}{=} \Delta_1 + \Delta_2. \end{aligned}$$

By recursion, we obtain

$$x_{t+k} = \phi^{t+m} x_{k-m} + \sum_{j=0}^{t-m-1} \phi^j u_{t+k-j},$$

$$\begin{aligned}
E(x_{s+k}^2|\mathcal{F}^{k-m}) &= \phi^{2(s-t)}E(x_{t+k}^2|\mathcal{F}^{k-m}) + (1-\phi^{2(s-t)})\sigma_u^2/(1-\phi^2), \\
E(x_{s+k}^2 x_{t+k}^2|\mathcal{F}^{k-m}) &= \phi^{2(s-t)}E(x_{t+k}^4|\mathcal{F}^{k-m}), \\
&\quad +(1-\phi^{2(s-t)})\sigma_u^2 E(x_{t+k}^2|\mathcal{F}^{k-m})/(1-\phi^2), \\
E(x_{t+k}^2|\mathcal{F}^{k-m}) &= \phi^{2(t+m)}x_{k-m}^2 + (1-\phi^{2(t+m)})\sigma_u^2/(1-\phi^2), \\
E(x_{t+k}^4|\mathcal{F}^{k-m}) &= \phi^{4(t+m)}x_{k-m}^4 + 6\phi^{2(t+m)}x_{k-m}^2\sigma_u^2(1-\phi^{2(t+m)})/(1-\phi^2) \\
&\quad +3\sigma_u^4(1-\phi^{4(t+m)})/(1-\phi^4) \\
&\quad +6\phi^2\sigma_u^4(1-\phi^{2(t+m)})(1-\phi^{2(t+m-1)})/((1-\phi^2)(1-\phi^4)),
\end{aligned}$$

$$\begin{aligned}
&E(x_{s+k}^2 x_{t+k}^2|\mathcal{F}^{k-m}) - \sigma_x^2[E(x_{s+k}^2|\mathcal{F}^{k-m}) + E(x_{t+k}^2|\mathcal{F}^{k-m})] + \sigma_x^4 \\
&= \phi^{2(s-t)}E(x_{t+k}^4|\mathcal{F}^{k-m}) - 2\phi^{2(s-t)}\sigma_u^2 E(x_{t+k}^2|\mathcal{F}^{k-m})/(1-\phi^2) + \phi^{2(s-t)}\sigma_u^4/(1-\phi^2)^2.
\end{aligned}$$

From the above equations, we know that

$$\begin{aligned}
\Delta_1 &= \frac{1}{T}\sum_{t=1}^{T}[E(x_{t+k}^4|\mathcal{F}^{k-m}) - 2\sigma_x^2 E(x_{t+k}^2|\mathcal{F}^{k-m}) + \sigma_x^4] \xrightarrow{\mathcal{L}_1} 2\sigma_u^4/(1-\phi^2)^2, \\
\Delta_2 &= \frac{2}{T}\sum_{t=1}^{T-1}\phi^{-2t}\Big[E(x_{t+k}^4|\mathcal{F}^{k-m}) - 2\sigma_u^2 E(x_{t+k}^2|\mathcal{F}^{k-m})/(1-\phi^2) \\
&\qquad +\sigma_u^4/(1-\phi^2)^2\,\Big]\sum_{s=t+1}^{T}\phi^{2s} \\
&= \frac{2}{T}\sum_{t=1}^{T-1}[E(x_{t+k}^4|\mathcal{F}^{k-m}) - 2\sigma_u^2 E(x_{t+k}^2|\mathcal{F}^{k-m})/(1-\phi^2) + \sigma_u^4/(1-\phi^2)^2] \\
&\qquad \times\phi^2(1-\phi^{2(T-t)})/(1-\phi^2) \\
&\xrightarrow{\mathcal{L}_1} 4\sigma_u^4\phi^2/((1-\phi^2)^3.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{T}E[(S_{k+T}-S_k)^2|\mathcal{F}^{k-m}] \xrightarrow{\mathcal{L}_1} 2\sigma_u^4(1+\phi^2)/(1-\phi^2)^3 = \sigma^2.$$

Theorem 1. Define

$$D_k = \frac{\sum_{t=1}^{k} x_t^2}{\sum_{t=1}^{T} x_t^2} - \frac{k}{T}, \quad k = 1, \ldots, T.$$

Then, under the same conditions as in Lemma 1,

$$B_T = C\sqrt{T}\max_{1\le k\le T}|D_k| \xrightarrow{\mathcal{D}} \sup_{0\le z\le 1}|B(z)|,$$

where $B(z)$ is the standard Brownian bridge and

$$C^2 = \frac{(1-\phi^2)}{2(1+\phi^2)}. \tag{2.1}$$

Proof. Define the partial sum of ξ_t as $S_k = \sum_{t=1}^{k} \xi_t$, where $\xi_t = x_t^2 - \sigma_x^2$. From the following equations

$$\begin{aligned} E\xi_0^2 &= 2\sigma_u^4/(1-\phi^2)^2, \\ E\xi_0\xi_k &= \phi^{2k} 2\sigma_u^4/(1-\phi^2)^2, \end{aligned}$$

we get

$$\sum_{k=1}^{\infty} |E\xi_0\xi_k| = 2\sigma_u^4/(1-\phi^2)^3 < \infty, \quad \text{since } |\phi| < 1.$$

Therefore,

$$\frac{1}{T}E(S_T)^2 \xrightarrow{\mathcal{P}} \sigma^2.$$

By Lemma 1, $\{(\xi_t, \mathcal{F}^t)\}$ is a mixingale of order $-1/2$. Since ξ_t is a strictly stationary process with constant finite second moment, ξ_t^2 is uniformly integrable. By Lemma 2, $E\{(S_{k+T} - S_k)^2 | \mathcal{F}^{k-m}\}/T \to \sigma^2$ in L_1 norm as $min(m, k, T) \to \infty$. Therefore, by Theorem 2.6 of McLeish(1975)

$$X_T(z) = \frac{1}{\sigma\sqrt{T}} S_{[Tz]} = \frac{1}{\sigma\sqrt{T}} \sum_{t=1}^{[Tz]} \xi_t \xrightarrow{\mathcal{D}} W(z),$$

where $W(z)$ is a standard Brownian motion. Moreover,

$$\begin{aligned} \sqrt{T}\left(\frac{1}{T}\sum_{t=1}^{T} x_t^2\right) D_k &= \frac{1}{\sqrt{T}} \sum_{t=1}^{k} (x_t^2 - \sigma_x^2) - \frac{k}{T}\frac{1}{\sqrt{T}} \sum_{t=1}^{T} (x_t^2 - \sigma_x^2) \\ &= \sigma(X_T(z) - zX_T(1)). \end{aligned}$$

By the law of large numbers, $\sum_{t=1}^{T} x_t^2/T \xrightarrow{\mathcal{P}} \sigma_x^2$ as $T \to \infty$. If we let $C_1 = \sigma_x^2/\sigma$ we obtain $C_1\sqrt{T}D_k \xrightarrow{\mathcal{D}} B(z)$ as $T \to \infty$.

We can obtain the information about the number and the locations of the change points by using the statistic D_k. Define

$$DT_k \stackrel{\text{def}}{=} \left(\frac{1}{T}\sum_{t=1}^{T} x_t^2\right) D_k = \frac{1}{T}\sum_{t=1}^{k} x_t^2 - \frac{k}{T}\frac{1}{T}\sum_{t=1}^{T} x_t^2, \tag{2.2}$$

where $k = [Tz]$, for $k = 1, \ldots, T$ and $0 \leq z \leq 1$. For example, assume that there is one change point $k^* = [Tz^*]$ in the series. Then,

$$
\begin{aligned}
DT_k &= \frac{1}{T}\sum_{t=1}^{k} x_t^2 - \frac{k}{T}\frac{1}{T}\sum_{t=1}^{T} x_t^2, \\
&\xrightarrow{\mathcal{P}} z\tau_1 - z(z^*\tau_1 + (1-z^*)\tau_2)) = z(1-z^*)\Delta, \quad \text{for} \quad k \leq k^*, \\
&\xrightarrow{\mathcal{P}} z^*\tau_1 + (z - z^*)\tau_2 - z(z^*\tau_1 + (1-z^*)\tau_2) = (1-z)z^*\Delta, \quad \text{for} \quad k > k^*,
\end{aligned}
$$

where $\Delta = \tau_1 - \tau_2$, $\tau_i = \sigma_{ui}^2/(1-\phi_i^2)$, and ϕ_i and σ_{ui}^2 are AR parameters and noise variances, respectively, for $i = 1, 2$. Therefore, DT_k is maximized at the change point k^*.

It is straight forward to show that the above results are still applicable to the MA(q) processes following Billingsley(1968).

3. SIMULATION

We compare the performances of two detection procedures, CUSUM based on DT_k in (2.2) and CUSUM-B suggested by Bai (1993), when a change occurs in the parameter of an AR(1) model. For simplicity, it is assumed that the change occurrs once in the middle of the series. For the values of ϕ_1, $(-.5, -.3, -.1, .1, .3, .5)$ are considered and ϕ_2 is chosen near ϕ_1. A large change is not considered since it may result in the model misspecification. Simulation results are summarized in Table 1. It is observed that CUSUM outperforms CUSUM-B in all the cases considered.

$$
\begin{aligned}
x_t &= \phi_1 x_{t-1} + u_t, \quad t = 1, \ldots, T/2 \\
&= \phi_2 x_{t-1} + u_t, \quad t = T/2+1, \ldots, T
\end{aligned}
$$

where $|\phi_i| < 1$ for $i = 1, 2$ and u_t follows an i.i.d. $N(0, 1)$.

4. REAL EXAMPLE

We apply the CUSUM procedure to the IBM closing stock prices, Series B in Box and Jenkins(1976). Many authors have used this example to evaluate the performance of their proposed statistics and algorithms. First, Box and

Table 1. Power comparison of CUSUM and CUSUM-B procedures

procedure	ϕ is changed from .1 to						sample size
	-.1	**.1 (null)**	.3	.5	.7	.9	
CUSUM	.0303	**.0279**	.0310	.0663	.1983	.3741	
CUSUM-B	.0635	**.0471**	.0338	.0215	.0129	.0036	T=100
CUSUM	.0391	**.0358**	.0463	.1534	.5649	.9747	
CUSUM-B	.0623	**.0462**	.0338	.0241	.0187	.0075	T=200
CUSUM	.0443	**.0418**	.0774	.3983	.9556	1.000	
CUSUM-B	.0673	**.0499**	.0371	.0263	.0240	.0312	T=500

procedure	ϕ is changed from .3 to						sample size
	-.1	.1	**.3 (null)**	.5	.7	.9	
CUSUM	.0379	.0300	**.0243**	.0378	.1091	.2085	
CUSUM-B	.0857	.0652	**.0455**	.0276	.0145	.0029	T=100
CUSUM	.0561	.0461	**.0318**	.0720	.3587	.9108	
CUSUM-B	.0883	.0656	**.0446**	.0294	.0173	.0053	T=200
CUSUM	.0860	.0733	**.0410**	.1789	.8351	.9998	
CUSUM-B	.0941	.0700	**.0496**	.0330	.0215	.0197	T=500

procedure	ϕ is changed from .5 to						sample size
	-.1	.1	.3	**.5 (null)**	.7	.9	
CUSUM	.0854	.0629	.0356	**.0191**	.0388	.0783	
CUSUM-B	.1274	.0965	.0688	**.0424**	.0204	.0038	T=100
CUSUM	.1865	.1483	.0730	**.0261**	.1189	.6002	
CUSUM-B	.1338	.0989	.0706	**.0447**	.0213	.0047	T=200
CUSUM	.4455	.3852	.1725	**.0357**	.3877	.9966	
CUSUM-B	.1431	.1086	.0770	**.0486**	.0251	.0132	T=500

procedure	ϕ is changed from -.1 to						sample size
	-.9	-.7	-.5	-.3	**-.1 (null)**	.1	
CUSUM	.3679	.1992	.0624	.0310	**.0275**	.0292	
CUSUM-B	.1341	.0992	.0771	.0606	**.0483**	.0370	T=100
CUSUM	.9776	.5679	.1513	.0478	**.0371**	.0388	
CUSUM-B	.1360	.1018	.0776	.0618	**.0471**	.0367	T=200
CUSUM	1.0000	.9560	.3961	.0729	**.0408**	.0447	
CUSUM-B	.1438	.1071	.0821	.0644	**.0495**	.0402	T=500

(Based upon 10000 replications)

Table 1. (continued)

	ϕ is changed from -.3 to						
procedure	-.9	-.7	-.5	**-.3 (null)**	-.1	.1	sample size
CUSUM	.2037	.1055	.0329	**.0257**	.0330	.0403	
CUSUM-B	.0997	.0758	.0583	**.0486**	.0397	.0323	T=100
CUSUM	.9111	.3631	.0741	**.0331**	.0461	.0542	
CUSUM-B	.1023	.0753	.0600	**.0469**	.0393	.0315	T=200
CUSUM	.9999	.8404	.1764	**.0381**	.0709	.0862	
CUSUM-B	.1076	.0800	.0632	**.0503**	.0411	.0335	T=500

	ϕ is changed from -.5 to						
procedure	-.9	-.7	**-.5 (null)**	-.3	-.1	.1	sample size
CUSUM	.0742	.0375	**.0201**	.0397	.0692	.0926	
CUSUM-B	.0752	.0584	**.0487**	.0418	.0359	.0317	T=100
CUSUM	.6047	.1140	**.0260**	.0722	.1503	.1899	
CUSUM-B	.0764	.0600	**.0475**	.0409	.0347	.0313	T=200
CUSUM	.9962	.3807	**.0355**	.1726	.3924	.4457	
CUSUM-B	.0795	.0629	**.0497**	.0424	.0368	.0335	T=500

(Based upon 10000 replications)

Jenkins (1976) fitted the ARIMA(0,1,1) model to this series and concluded that there is a change in the MA parameter which results in the inadequacy of the model. Wichern *et al.* (1976) applied their algorithm to the first difference of the series assuming that the series follows an AR(1) model and concluded that there are two innovation variance changes at $t = 180$ and $t = 235$. Their results are summarized in Table 4 of Whichern *et al.* (1976). They also claimed that there is some evidence that the AR parameter ϕ differ significantly from one another. Baufays and Rasson (1985) also argued that two variance changes occurred at $t = 235$ and $t = 279$. Tsay (1988) concluded that only one variance change occurred at $t = 237$. Inclán and Tiao (1994) used cumulative sums approach and found two variance changes at $t = 235$ and $t = 279$. They, however, analyzed the series under the assumption that the series follows a white noise process, i.e., i.i.d. $N(0, \sigma_u^2)$. Since the full series (sample size=369) follows a white noise process our test procedure is the same as the Inclán and Tiao (1994)'s and one variance change point at $t = 235$ is detected.

Table 2. Results of fitting an AR(1) model

periods	$\hat{\phi}_i$	$s.e$	t-value
full series(t= 1,. . . ,369)	.025	.0522	.48
first sub-series (t=1,. . . ,234)	.213	.0640	3.33
second sub-series (t=235,. . . ,369)	-.022	.0871	-.25

Hence, we partition the series into two sub-series : one from $t = 1$ to $t = 234$ and another from $t = 235$ to $t = 369$. From Table 2 we see that the first sub-series follows an AR(1) model but Inclán and Tiao (1994) ignore this point and apply their algorithm to this series as if it were a white noise process. When we apply the CUSUM procedure, we find no further variance change in the first sub-series since the value of the test statistic $B_{T1} = .9998$ is less than 1.36 (the 95th percentile of the standard Brownian bridege), where Figure 1 apparently shows that there is no dominant peak pattern in DT_k.

But, the second sub-series, which is assumed to follow a white noise process, has one variance change at $t = 279$ since

$$B_{T2} = 2.578488 \ > \ 1.36.$$

Figure 2 shows this more clearly. When we apply the detection procedure to the subseries, one from $t = 235$ to $t = 278$ and another from $t = 279$ to $t = 369$, no further change point is detected. In summary, we conclude that the IBM data has two change points at $t = 235$ and $t = 279$.

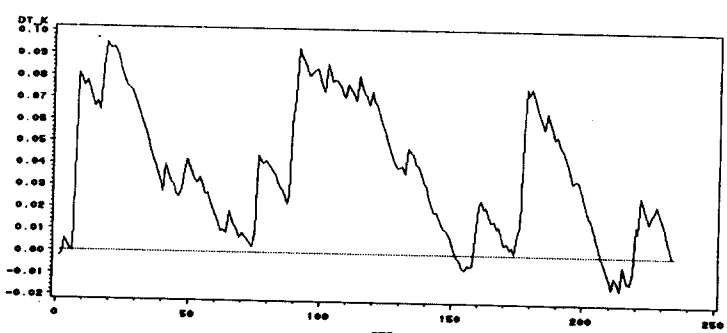

Figure 1. Plots of DT_k of the IBM data : First sub-period

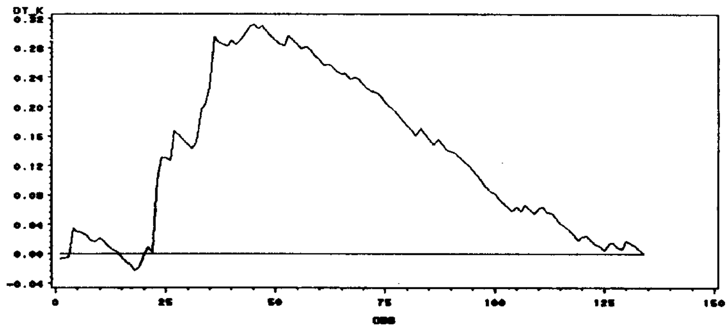

Figure 2. Plots of DT_k of the IBM data : Second sub-period

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