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Optimal Control of a Dam with a Compound Poisson Input†

Jiyeon Lee [1] and Eui Yong Lee [2]

Abstract

An infinite dam with a compound Poisson input having exponential jumps is considered. As an output policy, we adopt the P_λ^M policy. After assigning costs to the dam we obtain the long-run average cost per unit time of operating the dam and find the optimal values of λ and M which minimize the long-run average cost.

Key Words : Dam process; P_λ^M policy; Compound poisson process; Long-run average cost.

1. INTRODUCTION

In this paper, we consider an infinite dam with a compound Poisson input having exponential jumps and adop the P_λ^M policy as a control policy. It is assumed that the level of water is initially 0 and increases jumpwise due

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[1]Department of Statistics, Yeungnam University, Kyongsan 712-749, Korea.

[2]Department of Mathematics, Pohang University of Science and Technology, San 31 Hyoja-dong, Pohang 790-784, Korea.

to rains. The rains fall around the area of the dam according to a Poisson process with rate $\nu\,(>0)$ and each instataneously increases the level of water in the dam by an exponential amount with mean $\mu\,(>0)$. As an output policy, we consider the P_λ^M policy introduced by Faddy(1974). The release rate is 0 until the level of water reaches a threshold level $\lambda\,(>0)$. As soon as the level of water exceeds λ, the water starts to be released at a rate M until the dam becomes empty. Once the level of water reaches 0, the release rate turns to 0 until the level of water reaches λ again, and this cycle is repeated.

We assume that the release rate is increased from 0 to M with cost KM, K being a non-negative constant, but it is decreased from M to 0 with no cost and any such changes take effect instantaneously. We also assume that there are a reward of A monetary units per each output unit of water and a holding cost B per each unit of water in the dam, A, B being positive constants. Without loss of generality, we may put $A = 1$ by considering K and B as the relative costs compared to A. In section 2, we find the long-run average cost per unit time of operating the dam, assuming that the dam process $\{X(t),\, t \geq 0\}$ is stationary, where $X(t)$ denotes the level of water in the dam at time t. We calculate a unique value of λ which minimizes the long-run average cost and we also show that there is an optimal release rate M which minimizes the long-run average cost in section 3.

Faddy(1974) obtained the optimal release rate for a dam with finite capacity by assuming that the flow of water into the dam is determined by an almost surely continuous Wiener process. Zuckerman(1977) developed a rigorous approach to the problem considered by Faddy(1974) and also considered the optimal policy which minimizes the expected total discounted cost. A dam model with a compound Poisson input is introduced by Bar-Lev and Perry(1993). They derived the integral equations and solved the equations to obtain the stationary distribution of the level of water $X(t)$. In this paper, we extend the Bar-Lev and Perry's analysis by assigning costs to the dam and seeking to minimize the long-run average cost by varying the threshold level λ and the release rate M, when the amount of each input is exponentially distributed.

2. THE LONG-RUN AVERAGE COST FUNCTION

Consider the points where the reservoir becomes empty. The sequence of these points forms an embedded renewal process because after these points the process $\{X(t),\, t \geq 0\}$ regenerates itself. Let T be the random variable denoting the time between successive renewals. Then,

$$T \equiv T_{0\lambda} + T_{\lambda 0}, \tag{2.1}$$

where $\equiv$ denotes equality in distribution, $T_{0\lambda}$ is the period of waiting until the level of water exceeds the threshold λ after the reservoir being empty, and $T_{\lambda 0}$ is the period of releasing water until the dam becomes empty.

By the Renewal-Reward theorem, the long-run average cost per unit time $C(\lambda, M)$ for a given threshold level λ and release rate M is given by the expected cost per cycle divided by the average length of a cycle, where the cycle denotes the time between two successive embedded renewal points. Since the expected cost per cycle is $KM - ME[T_{\lambda 0}] + BE[\int_0^T X(t)dt)]$ it follows from (2.1) that the long-run average cost is given by

$$\begin{aligned} C(\lambda, M) &= \frac{KM - ME[T_{\lambda 0}] + BE[\int_0^T X(t)dt]}{E[T]} \\ &= \frac{KM - ME[T_{\lambda 0}]}{E[T_{0\lambda}] + E[T_{\lambda 0}]} + B\frac{E[\int_0^T X(t)dt]}{E[T_{0\lambda}] + E[T_{\lambda 0}]}. \end{aligned}$$

To obtain $E[T_{0\lambda}]$, we define $\{N(x),\, x \geq 0\}$ as a Poisson process with rate $1/\mu$, then we can see that $N(\lambda) + 1$ is the number of rains before water starts to be released and the waiting time $S_{N(\lambda)+1}$ in the Poisson process is the level of water at the moment that water starts to be released. Notice that

$$T_{0\lambda} \equiv E_1 + E_2 + \cdots + E_{N(\lambda)+1},$$

and

$$P\{S_{N(\lambda)+1} \leq x\} = 1 - \exp(-\frac{x-\lambda}{\mu}) \quad \text{for } x \geq \lambda,$$

where $E_1, E_2, \cdots$ are i.i.d. exponential random variables with rate ν. An argument similar to that of Lee and Lee(1993) shows that

$$E[T_{0\lambda}] = \frac{E[N(\lambda)+1]}{\nu} = \frac{\lambda + \mu}{\mu\nu}.$$

To obtain $E[T_{\lambda 0}]$ we apply the argument of Cox and Miller (1965, pp. 245-246) used to analyze the busy period of $M/G/1$ queue and can show that

$$E[T_{\lambda 0}] = \frac{E[S_{N(\lambda)+1}]}{M - \mu\nu} = \frac{\lambda + \mu}{M - \mu\nu},$$

when $M > \mu\nu$, i.e., when the release rate of water is larger than the input rate of water during the releasing period.

To calculate $E[\int_0^T X(t)dt]/E[T]$, we divide the original process $\{X(t),\ t \geq 0\}$ into the following two processes. Process $\{X_1(t),\ t \geq 0\}$ is formed by seperating from the original process the parts on the periods of releasing water and on the periods of the dam being empty and by connecting these parts together. Process $\{X_2(t),\ t \geq 0\}$ is formed by connecting the rest parts of the original process. Let T_1 and T_2 denote cycles between successive embedded renewal points in $\{X_1(t),\ t \geq 0\}$ and $\{X_2(t),\ t \geq 0\}$, respectively. Then we have that

$$\frac{E[\int_0^T X(t)dt]}{E[T]} = \frac{E[T_1]}{E[T]}\frac{E[\int_0^{T_1} X_1(t)dt]}{E[T_1]} + \frac{E[T_2]}{E[T]}\frac{E[\int_0^{T_2} X_2(t)dt]}{E[T_2]}. \tag{2.2}$$

Notice that

$$E[T_1] = E[T_{\lambda 0}] + 1/\nu = \frac{M + \lambda\nu}{\nu(M - \mu\nu)} \tag{2.3}$$

and

$$E[T_2] = E[T_{0\lambda}] - 1/\nu = \frac{\lambda}{\mu\nu}.$$

First, to obtain $E[\int_0^{T_1} X_1(t)dt]/E[T_1]$, we examine the process $\{X_1(t),\ t \geq 0\}$. Let $F_1(x,t) = P\{X_1(t) \leq x\}$ denote the distribution function of $X_1(t)$. In a small interval $(t, t+\Delta t)$, we can have the following mutually exclusive relations between $X_1(t)$ and $X_1(t+\Delta t)$ depending on whether rain comes;

$$X_1(t+\Delta t) = \begin{cases} 0, & \text{if } 0 \leq X_1(t) \leq M\Delta t \\ X_1(t) - M\Delta t, & \text{if } X_1(t) > M\Delta t, \end{cases}$$
$$\text{with probability } 1 - \nu\Delta t + o(\Delta t),$$

and

$$X_1(t+\Delta t) = \begin{cases} S_{N(\lambda)+1} - M\Delta t, & \text{if } X_1(t) = 0 \\ 0, & \text{if } 0 < X_1(t) \leq M\Delta t - Y \\ X_1(t) + Y - M\Delta t, & \text{if } X_1(t) > M\Delta t - Y, \end{cases}$$
$$\text{with probability } \nu\Delta t + o(\Delta t),$$

where Y is the exponential random variable with mean μ.

Thus

$$\begin{aligned} F_1(x, t+\Delta t) &= (1 - \nu\Delta t)P\{X_1(t) - M\Delta t \leq x\} \\ &\quad + \nu\Delta t P\{S_{N(\lambda)+1} - M\Delta t \leq x, X_1(t) = 0\} \\ &\quad + \nu\Delta t P\{X_1(t) + Y - M\Delta t \leq x, X_1(t) > 0\} + o(\Delta t). \end{aligned}$$

Performing a Taylor series expansion on $P\{X_1(t) - M\Delta t \le x\}$, rearranging the above equation and letting $\Delta t \to 0$, the following integro-differential equation is obtained;

$$\begin{aligned}\frac{\partial}{\partial t}F_1(x,t) &= M\frac{\partial}{\partial x}F_1(x,t) + \nu F_1(0,t)[1-\exp(-\frac{x-\lambda}{\mu})]I_{\{x\ge\lambda\}} \\ &\quad -\nu F_1(x,t) + \nu\int_0^x [1-\exp(-\frac{x-y}{\mu})]\frac{\partial}{\partial y}F_1(y,t)dy \\ &\quad \text{for } x \ge 0, \qquad (2.4)\end{aligned}$$

where I_A denotes the indicator of the event A.

Notice that $F_1(x,t)$ consists of a discrete probability $F_1(0,t)$ and a density $f_1(x,t)$ for $x > 0$, so does the stationary distribution $F_1(x)$. Notice also that this stationary distribution is the same as the limiting distribution $\lim_{t\to\infty} F_1(x,t)$, since the renewal points are embedded in the process $\{X_1(t),\ t \ge 0\}$ (Baxter and Lee(1987)).

Putting $\frac{\partial}{\partial t}F_1(x,t) = 0$ as t tends to ∞ and letting $\lim_{t\to\infty} f_1(x,t) = f_1(x)$ and $\lim_{t\to\infty} F_1(0,t) = F_1(0)$, we can derive from equation (2.4) that, for $x > 0$,

$$\begin{aligned}0 &= M\frac{d}{dx}f_1(x) + \frac{\nu}{\mu}F_1(0)\exp(-\frac{x-\lambda}{\mu})I_{\{x\ge\lambda\}} - \nu f_1(x) \\ &\quad +\frac{\nu}{\mu}\int_0^x \exp(-\frac{x-y}{\mu})f_1(y)dy, \qquad (2.5)\end{aligned}$$

and when $x = 0$,

$$0 = Mf_1(0) - \nu F_1(0). \qquad (2.6)$$

Taking Laplace transforms in equation (2.5), we have that

$$0 = M[sf_1^*(s) - f_1(0)] + \frac{\nu\exp(-\lambda s)}{1+\mu s}F_1(0) - \nu f_1^*(s) + \frac{\nu}{1+\mu s}f_1^*(s), \qquad (2.7)$$

where $f_1^*(s) = \int_0^\infty \exp(-sx)f_1(x)dx$.

Solving equation (2.7) with the normalizing condition $F_1(0) + f_1^*(0) = 1$ and equation (2.6), we have that

$$f_1^*(s) = \frac{\nu F_1(0)[1+\mu s - \exp(-\lambda s)]}{Ms(1+\mu s) - \mu\nu s},$$

where $F_1(0) = (M-\mu\nu)/(M+\lambda\nu)$, where $M > \mu\nu$.

Thus, the Laplace-Stieltjes transform of $F_1(x)$ is given by

$$\begin{aligned} F_1^*(s) &= \int_0^\infty \exp(-sx)dF_1(x) \\ &= F_1(0) + f_1^*(s) \\ &= \frac{F_1(0)[Ms(1+\mu s) + \nu - \nu\exp(-\lambda s)]}{Ms(1+\mu s) - \mu\nu s}. \end{aligned} \tag{2.8}$$

We now derive the first moment of the stationary distribution $F_1(x)$. Differentiating equation (2.8) with respect to s and letting $s \to 0$ yields

$$\lim_{t\to\infty} E[X_1(t)] = \frac{(M-\mu\nu)\nu\lambda^2 + 2M\mu\nu(\lambda+\mu)}{2(M-\mu\nu)(M+\lambda\nu)}, \quad \text{for } M > \mu\nu \tag{2.9}$$

which is the same as $E[\int_0^{T_1} X_1(t)dt]/E[T_1]$(see Wolff(1989, pp. 92)).

On the other hand, notice that

$$\int_0^{T_2} X_2(t)dt = E_1S_1 + \cdots + E_{N(\lambda)}S_{N(\lambda)},$$

and hence

$$E[\int_0^{T_2} X_2(t)dt] = \frac{\lambda^2}{2\mu\nu}. \tag{2.10}$$

Substituting expressions (2.3), (2.9) and (2.10) into equation (2.2) we obtain

$$\frac{E[\int_0^T X(t)dt]}{E[T]} = \frac{(M-\mu\nu)\lambda^2 + 2\mu^2\nu(\lambda+\mu)}{2(M-\mu\nu)(\lambda+\mu)} \quad \text{for } M > \mu\nu.$$

In summary, the long-run average cost is given by

$$C(\lambda, M) = \frac{K\mu\nu(M-\mu\nu)}{\lambda+\mu} - \mu\nu + B\frac{(M-\mu\nu)\lambda^2 + 2\mu^2\nu(\lambda+\mu)}{2(\lambda+\mu)(M-\mu\nu)}, \quad \text{for } M > \mu\nu.$$

3. OPTIMAL POLICY

We, now, try to find the optimal value of the threshold λ and that of the output rate M which minimize the long-run average cost $C(\lambda, M)$ obtained in section 2. To find the optimal value λ^* of λ for a given $M > \mu\nu$, we differentiate $C(\lambda, M)$ with respect to λ and put $\frac{\partial}{\partial\lambda}C(\lambda, M) = 0$ which is equivalent to

$$B\lambda^2 + 2B\mu\lambda - 2K\mu\nu(M - \mu\nu) = 0.$$

Then the above equation shows that λ^* is given by

$$\lambda^* = -\mu + \sqrt{\mu^2 + \frac{2K\mu\nu(M - \mu\nu)}{B}}\ (> 0). \tag{3.1}$$

For a given $\lambda > 0$, to find the optimal value M^* of M, we again differentiate $C(\lambda, M)$ with respect to M and put $\frac{\partial}{\partial M}C(\lambda, M) = 0$ which is equivalent to

$$KM^2 - 2K\mu\nu M + K\mu^2\nu^2 - B\mu(\lambda + \mu) = 0. \tag{3.2}$$

Then, the above equation shows that M^* is given by

$$M^* = \mu\nu + \sqrt{\frac{B\mu(\lambda + \mu)}{K}}.$$

From equations (3.1) and (3.2) we can see that as the releasing cost K increases, the optimal value λ^* increases, but the optimal value M^* decreases, so that we open the dam less-frequently. On the other hand, if the holding cost B increases then M^* increases but λ^* decreases, so that less amount of water is kept in the dam in the long-run.

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