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Nonparametric Tests for Monotonicity Properties of Mean Residual Life Function†

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Abstract

This is primarily an expository paper that presents several nonparametric procedures for testing exponentiality against certain monotonicity properties of the mean residual life function. In addition to the monotone behavior of mean residual life function, tests against the trend change in such function attract a great deal of attention of late in reliability analysis. In this note, we present some of the known testing procedures regarding the behavior of mean residual life function. These tests are also compared in terms of asymptotic relative efficiency and empirical power against a few alternatives. The tests based on incomplete data are also briefly discussed.

Key Words : L-statistic; DMRL(IMRL); IDMRL(DIMRL); U-statistic; Asymptotic relative efficiency; Randomly censored data.

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1. INTRODUCTION

Let X be a nonnegative absolutely continuous random variable(often describing the life of a component)with the life distribution $F(x)$ and let $\bar{F}(x) = 1 - F(x)$. The conditional expectation

$$m(t) = E(X - t \mid X > t) = \int_t^\infty \bar{F}(x)dx \Big/ \bar{F}(t)$$

is defined as a mean residual life function. It is called the expectation of life at t in survival analysis. The concept of mean residual life has been investigated extensively in the literature and find vast applications in the various areas of practical interest. See, e.g., Hollander and Proschan(1984), Kotz and Shanbhag(1980), Guess and Proschan(1988), and the bibliography therein.

The mean residual life was first introduced by Waston and Wells(1961). They use the mean residual life for studying the concept of burn-in. Since then the mean residual life has been used not only for parametric modeling, but also for nonparametric modeling. Hall and Wellner(1981) discuss parametric uses of the mean residual life. Based on the behavior of mean residual life function, several nonparametric classes of life distributions have been defined and studied by many reliability scientists. Barlow and Proshcan(1965) introduce the decreasing mean residual life(DMRL) class which consists of life distributions whose mean residual life functions are non-increasing, and they note that DMRL class arises naturally in reliability theory. The class of life distributions with non-decreasing mean residual life function is called increasing mean residual life(IMRL). Besides the class of life distributions with $m(t) \leq m(0)$ for $t \geq 0$ is defined as new better than used in expection(NBUE). When $m(t) \geq m(0)$ for $t \geq 0$ the class is called new worse than used in expectation(NWUE).

The DMRL class models aging that is adverse. That is, the older a DMRL unit is, the shorter is the remaining life of the unit on the average. Regarding IMRL class, Morrison(1978) mentions that the IMRL distributions are found to be useful as models in the social sciences for the lifelengths of wars and strikes. Brown(1983) mentions the areas in which the IMRL class plays an important role and studies approximating IMRL distributions by exponentials.

Guess, Hollander and Proschan(1986) introduce two nonparametric classes of life distributions which show the trend change in mean residual life. One such class consists of those with increasing initially, then decreasing mean residual life and is called IDMRL. The other class is called DIMRL, with decreasing initially and then increasing mean residual life. The lifelengths of humans can be reasonably considered as IDMRL model. High infant mortality explains the initial IMRL phase. Aging and deterioration explain the later DMRL state. More examples and discussions for such trend changes in mean residual life are given in Guess(1984).

This article presents several nonparametric methods for testing the null hypothesis that the underlying life distribution is exponential against the alternative hypothesis that the underlying life distribution follows a certain monotonicity property of the mean residual life function. In Section 2 several tests for detecting monotone behavior of mean residual life are discussed. Section 3 considers testing for trend change in mean residual life. Brief discussion for the case of censored data is given in Section 4.

2. TESTS FOR MONOTONE MEAN RESIDUAL LIFE

Let $X_1, \cdots, X_n$ be a random sample from F and we let F_n to be the empirical distribution function obtained from the sample. Based on the random sample, the problem of our interest is to test

$$H_0 : F \text{ is the exponential distribution}$$
$$\text{(i.e. } \bar{F}(x) = \exp(-x/\mu), x \geq 0, \text{ with } \mu \text{ unknown)}$$

against

$$H_1 : F \text{ is DMRL, but is not exponential.}$$

Hollander and Proschan(1975) Test

The first nonparametric method dealing with this problem is proposed by Hollander and Proschan(1975)(HP(1975)). Their test statistic is motivated by considering the following parameter as a measure of deviation. Let

$$\Delta(F) = \int\int_{s<t} \bar{F}(s)\bar{F}(t)(m(s) - m(t))dF(s)dF(t). \qquad (2.1)$$

Under H_0, $\Delta(F) = 0$ and under H_1, $\Delta(F) > 0$ assuming that F is continuous. The integrand of $\Delta(F)$ is a weighted measure of the deviation from H_0 towards H_1, and $\Delta(F)$ is an average of this deviation. The weights $\bar{F}(s)$ and $\bar{F}(t)$ represent the proportions of the population still alive at s and t, respectively and thus provide comparisons concerning the mean residual lifelengths from s and t, respectively. Substituting the empirical distribution function and deriving an asymptotically equivalent statistic to $\Delta(F_n)$, HP(1975) obtain

$$V = n^{-4}\sum_{i=1}^{n} C_{in}X_{(i)}, \tag{2.2}$$

$$C_{in} = \frac{4}{3}i^3 - 4ni^2 + 3n^2 i - \frac{1}{2}n^3 + \frac{1}{2}n^2 - \frac{1}{2}i^2 + \frac{1}{6}i, \tag{2.3}$$

where $X_{(1)} < \cdots < X_{(n)}$ are the order statistics of the sample and $\bar{X}$ is the sample mean. From (2.2) and (2.3), V can be approximated by

$$V \cong n^{-1}\sum_{i=1}^{n} J_1(\frac{i}{n})X_{(i)}, \tag{2.4}$$

where

$$J_1(u) = \frac{4}{3}u^3 - 4u^2 + 3u - \frac{1}{2}$$

HP(1975) use $V^* = V/\bar{X}$, where V is defined in (2.4), as their test statistic. V^* is scale invariant and is a L-statistic. Thus, by applying the L-statistic theory the asymptotic normality of the test statistic V^* is established. HP's(1975) large sample α-level test for H_0 versus H_1 is to reject H_0 in favor of H_1 if $(210n)^{1/2}V^* \geq z_\alpha$, where z_α is the upper α-quantile of standard normal distribution. Analogously, H_0 is rejected in favor of $H_1^{'}$: F is IMRL, but is not exponential if $(210n)^{1/2}V^* \leq -z_\alpha$. The consistency of such tests follows from the fact that $\Delta(F) > (<) 0$ under $H_1(H_1^{'})$ if F is continuous.

Bergman and Klefsjö(1989) Test

Bergman and Klefsjö(1989)(BK(1989)) develop a family of test statistics, intended for testing a DMRL, which includes the test statistic by HP(1975) as a special case. They generalize the idea by HP's(1975) by using the weights $\bar{F}^j(s)\bar{F}^k(t)$, instead of $\bar{F}(s)\bar{F}(t)$, where j and k are positive integer values. In this case, the measure of DMRL-ness of F is

$$\Delta_{jk}(F) = \int\int_{s\ t} \bar{F}^j(s)\bar{F}^k(t)(m(s) - m(t))dF(s)dF(t).$$

The asymptotic normality of the test statistic $\Delta_{jk}(F_n)/\bar{X}$ is obtained by using the L-statistic theory and its efficacy values are calculated for several DMRL alternatives.

Aly(1990) Test

As a measure of deviation of F from H_0 in favor of H_1, Aly(1990) considers the parameter

$$r(F) = \int_0^1 h(y) dF^{-1}(y), \tag{2.5}$$

where

$$h(y) = (1-y)(1+\log(1-y)), \qquad 0 \le y \le 1.$$

It is shown that $r(F)$ is equivalent to

$$r(F) = \int_0^\infty (m(0) - m(t)) dF(t).$$

$r(F)$ is motivated by the fact that if F is DMRL, $\bar{F}(t)m'(t) \le 0$ for all t and it turns out that $r(F)$ is also a measure of deviation of F in favor of NBUE alternatives. The proposed test statistic is based on

$$r(F_n) = (1 + n^{-1} \sum_{i=1}^{n} C_{in} X_{(i)}) \Big/ \bar{X},$$

where $C_{nn} = -\log n$ and $C_{in} = \log((n-i+1)/n)((n-i+1)/(n-i))^{n-i}, i = 1, \cdots, n-1$. The asymptotic null distribution of $A = n^{1/2} r(F_n)$ is proved to be a standard normal distribution. Thus, Aly's(1990) test is to reject H_0 in favor of H_1 if $A \ge z_\alpha$ and is consistent against all continuous NBUE(and thus against all continuous DMRL) alternatives.

Bandyopadhyay and Basu(1990) Test

The test is motivated by the observation that F belongs to the class of DMRL(IMRL) distributions if and only if

$$m(kx) \ge (\le) m(x)$$

for all $0 < k < 1$ and all $x \ge 0$. They consider the parameter

$$B(F;k) = \int_0^\infty \bar{F}(x)\bar{F}(kx)(m(kx) - m(x)) dF(x). \tag{2.6}$$

$B(F;k)$ measures the deviation of H_0 towards $H_1(H_1')$. Under H_0, $B(F;k) = 0$ and under $H_1(H_1')$, $B(F;k) > (<) 0$. The weights $\bar{F}(x)$ and $\bar{F}(kx)$ in the integrand of (2.7) represent the proportions of the population still alive at x and kx, respectively. A natural test statistic is obtained by replacing F of (2.7) by F_n. Instead of $B(F_n;k)$, Bandyopadhyay and Basu(1990)(BB(1990)) use the following U-statistic which is asymptotically equivalent to $B(F_n;k)$.

$$U_n^*(k) = (n(n-1)(n-2))^{-1} \sum \Phi_k(X_i, X_j, X_k), \tag{2.7}$$

where $\sum$ is the sum taken over all permutations (i,j,k) of 3 distinct integers chosen from $1, \cdots$, n , $n \geq 3$, with

$$\Phi_k(a,b,c) = (a-kc)I(a-kc)I(b-c) - (a-c)I(a-c)I(b-kc)$$

and $I(x) = 1$ or 0 according as $x > 0$ or $x \leq 0$. To make the test statistic scale invariant, they use

$$U_n(k) = U_n^*(k)/\bar{X}.$$

The asymptotic normality of $U_n(k)$ is established by using the U-statistic theory and the test is proved to be consistent against DMRL alternatives.

Ahmad(1992) Test

The test proposed by Ahmad(1992) is motivated by a simple observation that if the mean residual life function, $m(t)$, is differentiable and $m(t)$ is decreasing, then $m'(t) = dm(t)/dt \leq 0$ for $t \geq 0$. Let f denote the probability density function corresponding to F and let $\nu(x) = \int_x^\infty \bar{F}(u)du$. Then the deviation of F from H_0 towards H_1 can be measured by the parameter

$$\begin{aligned} \delta(F) &= \int_0^\infty [\bar{F}^2(x) - f(x)\nu(x)]dx \\ &= \int_0^\infty [2x\bar{F}(x) - \nu(x)]dF(x), \end{aligned} \tag{2.8}$$

where the second equality is obtained by integration by parts assuming that $\lim_{x\to\infty} x\bar{F}^2(x) = 0$. The integrand in the middle expression of (2.8) is positive if and only if $m(t)$ is decreasing. Thus, the large values of $\delta(F)$ would support H_1 and $\delta(F) = 0$ if and only if F is an exponential distribution. Ahmad's(1992) test statistic, $\delta(F_n)$, turns out to be an U-statistic and has the following expression.

$$\delta(F_n) = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} (3X_i - X_j) I(X_j - X_i),$$

where $I(a) = 0$ if $a \leq 0$ and $I(a) = 1$ if $a > 0$. The asymptotic null distribution of $U = n^{1/2}\delta(F_n)$ is normal with mean 0 and variance of $\frac{1}{3}$. Thus, Ahmad's(1992) large sample test for H_0 versus H_1 is to reject H_0 in favor of H_1 if $\sqrt{3}U \geq z_\alpha$. For H_0 versus $H_1^{'}$, H_0 is rejected in favor of $H_1^{'}$ if $\sqrt{3}U \leq -z_\alpha$.

Efficiency Comparison of the Tests

Let V, A, and U denote HP(1975) test, Aly(1990) test and Ahmad(1992) test, respectively. To compare these tests in terms of Pitman asymptotic relative efficiencies(ARE), we consider the following three DMRL distributions.

$$\bar{F}_1(x) = \exp(-(x + \tfrac{1}{2}\theta x^2)), \theta \geq 0, x \geq 0 : \text{ Linear Failure Rate}$$
$$\bar{F}_2(x) = \exp(-(x + \theta(x + e^{-x} - 1))), \theta \geq 0, x \geq 0 : \text{ Makeham}$$
$$\bar{F}_3(x) = \exp(-x^\theta), \theta \geq 1, x \geq 0 : \text{ Weibull}$$

Aly(1990) and Ahmad(1992) show that $ARE_{F_1}(A,V)$=1.219, $ARE_{F_2}(A,V)$= 1.0714, $ARE_{F_3}(A,V)$=1.4272, $ARE_{F_1}(U,V)$ =2.63, $ARE_{F_2}(U,V)$ =4.2, and $ARE_{F_3}(U,V)$=1.43. From these ARE values we can obtain $ARE_{F_1}(A,U) =$ 0.463, $ARE_{F_2}(A,U)$=0.255, and $ARE_{F_3}(A,U) = 0.998$. This indicates that both Aly's test and Ahmad's test outperform HP's test and Ahmad's test works better than Aly's test.

BB(1990) and BK(1989) present the ARE values of their tests with respect to the HP's test for various choices of k and (j,k). A few selected ARE values for both tests are given in Table 1. Here V, B_k and V_{jk} denote HP(1975) test, BB(1990) test and BK(1989) test, respectively.

Table1. ARE Values of B_k test and V_{jk} test with respect to V test.

		$ARE(B_k, V)$				$ARE(V_{jk}, V)$			
Alternative	k	0.000001	0.01	0.05	(j,k)	(2, 1)	(2, 5)	(3, 4)	(5, 5)
F_1		0.914	0.913	0.911		0.988	0.548	0.598	0.506
F_2		1.429	1.418	1.379		1.173	1.252	1.225	1.113
F_3		2.058	1.948	1.734		1.213	1.398	1.431	1.495

The values of k and (j,k) listed for both tests represent the cases for which BB test and BK test perform best against each alternative distribution. For

BK test, V_{21}, V_{25} and V_{55} tests are the best for F_1, F_2 and F_3 alternatives, respectively. BB(1990) shows that their test is the best for a very small value of k, such as $k = 0.000001$. However, for practical considerations, one might choose $k = 0.01$ or $k = 0.05$ since these tests achieve reasonably high ARE's with respect to other tests. Overall, BB test and BK test perform slightly better than HP test against F_2 and F_3, but not as effective as the HP test against F_1.

3. TESTS FOR TREND CHANGE IN MEAN RESIDUAL LIFE

In this section we present some of the nonparametric tests for detecting the trend change in mean residual life. Following the introduction of IDMRL(DIMRL) class by Guess, Hollander and Proschan(1986), a great deal of researches have been focused on the trend change not only in mean residual life but also in failure rate. Although many well-known life distributions show monotone failure rate or mean residual life, there exist several situations for which the monotonicity assumption does not model aging properties properly. The lognormal, inverse Gaussian and Hjorth distributions are examples which exhibit such trend changes. Klefsjö(1988) also considers the trend change in the NBUE-property.

The following tests have been proposed for testing H_0 versus each of the alternatives

$$H_2 : F \text{ is IDMRL}(\tau)\text{, but is not exponential,}$$
$$H_3 : F \text{ is IDMRL}(p)\text{, but is not exponential.}$$

Under H_2, the turning point(at which the trend change of mean residual life occurs), τ, is known and under H_3, the proportion, p, of the population that dies at or before the turning point is known (knowledge of τ itself is not assumed).

Guess, Hollander and Proschan(1986) Test

For both H_2 and H_3, they consider the following parameter

$$\begin{aligned} T(F) &= \int_0^{\tau} \int_0^t \bar{F}(s)\bar{F}(t)[m(t) - m(s)]dF(s)dF(t) \\ &\quad + \int_{\tau}^{\infty} \int_{\tau}^t \bar{F}(s)\bar{F}(t)[m(s) - m(t)]dF(s)dF(t). \qquad (3.1) \end{aligned}$$

Against H_3 for which p is known instead of τ, we may use $\tau = F^{-1}(p)$. Under H_0, $T(F) = 0$ and under H_2 or H_3, $T(F) > 0$. Thus, larger values of $T(F)$ support H_2 or H_3. A natural test statistic to consider is $T_n = T(F_n)$. Let

$$\sigma^2(T,F) = \mu^2\left[-\frac{1}{15}F^5(\tau) + \frac{1}{6}F^4(\tau) - \frac{1}{6}F^3(\tau) + \frac{1}{10}F^2(\tau) - \frac{1}{30}F(\tau) + \frac{1}{210}\right],$$

and

$$\sigma^2(p) = -\frac{1}{15}p^5 + \frac{1}{6}p^4 - \frac{1}{6}p^3 + \frac{1}{10}p^2 - \frac{1}{30}p + \frac{1}{210},$$

where $\mu = \int_0^\infty \bar{F}(x)dx$ is the population mean. To establish asymptotic normality of T_n, they use the differentiable statistical function approach of von Mises(1947) and the L-statistic theory for H_2 and H_3, respectively. Against H_2, the IDMRL test is to reject H_0 in favor of H_2 at the approximate level α if $\tilde{T}_n = n^{1/2}T(F_n)/\hat{\sigma}_n \geq z_\alpha$ where $\hat{\sigma}_n^2 = \sigma^2(T, F_n)$. The DIMRL test rejects H_0 in favor of H_2' : F is DIMRL(τ), but is not exponential at the approximate level α if $\tilde{T}_n \leq -z_\alpha$. Against H_3, the IDMRL(p) test rejects H_0 in favor of H_3 if $\tilde{V}_n = n^{1/2}V_n^*/\sigma(p) \geq z_\alpha$. If $\tilde{V}_n \leq -z_\alpha$, then H_0 is rejected in favor of H_3' : F is DIMRL(p), but is not exponential. Here $V_n^* = V_n/\bar{X}$ and V_n is a modified version of $T(F_n)$. For more details see Guess, Hollander and Proschan(1986).

Aly(1990) Test

As a measure of deviation of F from H_0 for each alternative, Aly(1990) considers the parameter

$$\begin{aligned}\Delta(p;F) \;=\; & \int_0^p [(1-y)r(F^{-1}(y))m(F^{-1}(y)) - (1-y)]dF^{-1}(y) \\ & - \int_p^1 [(1-y)r(F^{-1}(y))m(F^{-1}(y)) - (1-y)]dF^{-1}(y).\end{aligned} \quad (3.2)$$

$\Delta(p;F)$ is motivated by the fact that $\bar{F}(t)m'(t) \geq 0$ on $[0,\tau)$ and ≤ 0 on $[\tau,\infty)$ under H_2 and $m'(t) = h(t)m(t) - 1$, where $h(t)$ is a failure rate function. Note that $p = F(\tau)$. For testing H_0 against H_2, $T_1(\tau) = \Delta(F_n(\tau);F_n)/\bar{X}$ is used as its test statistic. For testing H_0 against H_3, the proposed test statistic $T_2(p)$ is obtained by $\Delta(p;F_n)/\bar{X}$.

Aly(1990) also proposes a test for H_0 versus

H_4 : F is IDMRL, but is not exponential.
(neither p nor τ is known)

The proposed test statistic is

$$\begin{aligned} T_3 &= \sup_{0<p<1} n^{1/2}\Delta(p;F_n)/\bar{X} \\ &\approx \max_{1\ j\leq n-1} n^{1/2}(2A(j)-A(1))/\bar{X}, \end{aligned}$$

where for $j = 1,\cdots,n$

$$A(j) = \sum_{i=j}^{n}(1-\frac{i-1}{n})(1+\log\frac{n-i+1}{n-j+1})(X_{(i)}-X_{(i-1)}).$$

The percentage points of each test statistic are calculated by a Monte Carlo simulation study to obtain the empirical critical values for each test.

Lim and Park(1995) Test

The test statistic for H_0 versus H_3 is motivated by Ahmad's(1992) test for DMRL alternatives, which uses the first derivative of mean residual life to measure the DMRL-ness. Assuming that the proportion, p, of the population that dies at or before the change point of mean residual life is known, the following parameter is considered to derive the test statistic.

$$\begin{aligned} L(F) &= \int_0^{F^{-1}(p)}[f(x)\nu(x)-\bar{F}^2(x)]dx + \int_{F^{-1}(p)}^{\infty}[\bar{F}^2(x)-f(x)\nu(x)]dx \\ &= \int_0^{F^{-1}(p)} x(1-4\bar{F}(x))dF(x) \\ &\qquad + \int_{F^{-1}(p)}^{\infty} x(4\bar{F}(x)-1+2p)dF(x) - 2(1-p)^2F^{-1}(p) \\ &= \int_0^{\infty} xJ(F(x))dF(x) - 2(1-p)^2F^{-1}(p), \end{aligned}$$

where $\nu(x) = \int_x^\infty \bar{F}(u)du$ and

$$J(u) = \begin{cases} 1-4(1-u) & \text{for } 0 \leq u < p \\ 4(1-u)-1+2p & \text{for } p \leq u < 1. \end{cases}$$

The test statistic, $L(F_n)$, results in a L-statistic and its asymptotic normality can be established by the L-statistic theory. Consequently, it follows that the asymptotic null distribution of $\tilde{L_n} = (3n)^{1/2}L(F_n)/\bar{X}$ is a standard normal distribution. Thus, a large sample approximate α-level test of H_0 versus H_3 is to reject H_0 if $\tilde{L}_n \geq z_\alpha$. For testing H_0 versus H_3', H_0 is rejected if $\tilde{L}_n \leq -z_\alpha$. Lim and Park(1995) prove the consistency of the proposed test and conduct Monte Carlo simulation studies to investigate the speed of convergence of the test statistic to normality.

Empirical Powers of the Tests

To compare three tests discussed above, Lim and Park(1995) calculate empirical powers of each test against lognormal alternatives for various choices of n and p. The values in Table 2 are the empirical powers of three tests based on 1000 simulations, each time with a sample of size n. We reproduce a part of the results from Lim and Park(1995). Here V_n, U_n and T_n denote Guess, Hollander and Prochan's test, Aly's test and Lim and Park's test, respectively.

Table 2. Empirical Power of V_n, U_n, T_n test vs Lognormal Altenative ($\mu = 0$ and $\sigma > 0$) for $\alpha = 0.05$.

		n				
$\sigma(p)$	test statistic	10	20	40	60	80
0.4(0.9624)	V_n	0.391	0.330	0.348	0.460	0.566
	U_n	0.340	0.969	1.000	1.000	1.000
	T_n	0.893	1.000	1.000	1.000	1.000
0.8(0.4977)	V_n	0.304	0.211	0.167	0.151	0.185
	U_n	0.112	0.189	0.387	0.509	0.658
	T_n	0.042	0.259	0.596	0.780	0.888
1.6(0.0736)	V_n	0.539	0.737	0.957	0.990	0.998
	U_n	0.190	0.525	0.877	0.961	0.992
	T_n	0.615	0.818	0.946	0.994	0.995

Table 2 shows that all three tests are more effective in detecting the trend change when p is very small and very large. When p is close to 0.5, all three tests exhibit poor powers even when n is relatively large. The table also shows that Lim and Park's(1995) test achieves the hightest power among three tests in most cases. More extensive tables are given in Lim and Park(1995).

4. GENERALIZATIONS TO CENSORED CASE

This section discusses the tests based on randomly right censored data. The basic idea is to replace F in the expression of the measure of deviation of F from H_0 by the Kaplan-Meier estimator given in (4.1) below.

Let $X_1, \cdots, X_n$ be independent, identically distributed (i.i.d.) according to a continuous life distribution F and let $Y_1, \cdots, Y_n$ be i.i.d. according to a continuous censoring distribution H. Here we assume that X's are independent of Y's. The censoring distribution H is assumed to be unknown and is treated as a nuisance parameter. In the randomly right censored model, the pairs (Z_i, δ_i), $i = 1, \cdots, n$, where $Z_i = min(X_i, Y_i)$ and $\delta_i = 1$ if $Z_i = X_i$, or $= 0$ if $Z_i = Y_i$ are observed.

In the randomly censored model, Kaplan and Meier(1958) propose the product limit estimator

$$\bar{F_n}^c(t) = \Pi_{\{i:Z_{(i)} \leq t\}}((n-i)/(n-i+1))^{\delta_{(i)}}, \qquad t \in [0, Z_{(n)}], \tag{4.1}$$

where $Z_{(0)} = 0 < Z_{(1)} < \cdots < Z_{(n)}$ are the ordered Z's and $\delta_{(i)}$ is the δ corresponding to $Z_{(i)}$. Here we treat $Z_{(n)}$ as an uncensored observation, whether it is uncensored or censored. When censored observations are tied with uncensored observations, we follow the standard convention of treating the uncensored observations as preceding the censored observations.

Chen, Hollander and Langberg(1983)(CHL(1983)) generalize Hollander and Proschan's(1975) test to accomodate the randomly right censored data. Their test statistic is derived by replacing $\bar{F}$ of (2.1) by the Kaplan-Meier estimator, $\bar{F_n}^c$. Lim and Park(1993)(LP(1993)) use Ahmad's(1992) test to generalize to the censored case and propose a new test, which is a competitor to CHL(1983) test. LP(1993) obtain their test statistic by replacing $\bar{F}$ of (2.6) by $\bar{F_n}^c$. The asymptotic normality of each of CHL(1983) and LP(1993) test statistics is established similarly by applying the techniques of Joe and Proschan(1982) and Gill(1983). Both tests are proved to be consistent against DMRL alternatives.

Table 3. Asymptotic Relative Efficiency of LP test with respect to CHL test when $\bar{H}(x) = e^{-\lambda x}$, $x \geq 0$

λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
F_1	0.914	0.983	1.059	1.140	1.227	1.317	1.410	1.503	1.596	1.688
F_2	1.429	1.536	1.655	1.782	1.917	2.058	2.203	2.349	2.494	2.637
F_3	2.057	2.212	2.383	2.566	2.761	2.964	3.172	3.382	3.591	3.798

LP(1993) compare CHL(1983) test and LP(1993) test by calculating Pitman asymptotic relative efficiencies for three DMRL alternatives, which are given in Section 2. For completeness of our presentation on comparision of these two tests, we reproduce the results of LP(1993) in Table 3. Table 3 presents the ARE values of LP test with respect to CHL test when the censoring distribution is exponential, $\bar{H}(x) = 1$, for $x < 0$, $\bar{H}(x) = exp(-\lambda x)$, for $x \geq 0$. For this choice of H, the restriction $\lambda < 1$ must be imposed to establish the asymptotic normality of its test statistic. See CHL(1983) for the detailed discussions on this condition.

Table 3 indicates that LP(1993) test compares favorably with CHL(1983) test in all cases considered, except when $\lambda = 0.0$ and $\lambda = 0.1$ for F_1. In these cases the CHL test exhibits slightly higher efficiencies than the LP test. It also shows that LP(1993) test has relatively higher efficiencies with regard to CHL(1983) test for large values of λ, which is the case when heavy censoring occurs. Note that $\lambda = 0$ corresponds to no cencoring. Bergman and Klefsjö(1989)(BK(1989)) also discuss the censored case along with uncensored case against DMRL alternatives. Their test is a generalization of CHL(1983) test. Table 4 provides the values of ARE of BK(1989) test with respect to CHL(1983) test when λ =0.1, 0.5, and 0.9 and the censoring distribution is exponential. When λ =0(that is, no censoring), the ARE values of BK test with respect to CHL test, which is equivalent to HP test, are given in Table1. These ARE values in Table 4 are computed using the squared efficacy values of BK test given in BK(1989).

Table 4.ARE Values of BK test with respect to CHL test when $\bar{H}(x) = e^{-\lambda x}$, $x \geq 0$

	$\lambda = 0.1$			$\lambda = 0.5$			$\lambda = 0.9$		
(j,k)	F_1	F_2	F_3	F_1	F_2	F_3	F_1	F_2	F_3
$(2,1)$	1.002	1.087	1.149	1.064	1.160	1.229	1.111	1.231	1.333
$(2,5)$	0.627	1.313	2.491	1.246	2.611	4.943	3.000	6.231	12.333
$(3,4)$	0.679	1.345	2.368	1.266	2.507	4.429	2.889	5.615	10.333
$(5,5)$	0.585	1.288	2.711	1.212	2.674	5.629	3.111	6.769	15.000

Table 4 indicates that when λ is small (light censoring occurs), CHL test is more effective against F_1. However, as λ increases(the amount of censoring increases) BK test outperforms CHL test in all cases. The same combinations of (j,k) as in Table 1 are listed for illustrative purposes.

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