H^∞ Control for Linear Systems with Delayed State and Control

Eun-Tae Jeung, Do-Chang Oh, and Hong-Bae Park

Abstract

This paper presents an H^∞ controller design method for linear time-invariant systems with delayed state and control. Using the second method of Lyapunov, the stability for delayed systems is discussed. For delayed systems, we derive a sufficient condition of the bounded real lemma(BRL) which is similar to BRL for nondelayed systems. And the sufficient conditions for the existence of an H^∞ controller of any order are given in terms of three linear matrix inequalities(LMIs). Furthermore, we briefly explain how to construct such controllers from the positive definite solutions of their LMIs and give a simple example to illustrate the validity of the proposed design procedure.

I. Introduction

Since 1980s, the H^∞ control problem has been extensively studied. It is well known that the state-space result of Doyle *et al.*[1] is an efficient and numerically good method for the standard H^∞ control problem. The existence conditions for an H^∞ controller were described by two Riccati equations and a spectral radius condition. Gahinet *et al.*[2] and Iwasaki *et al.*[3] extended the standard H^∞ control problem to the general H^∞ control problem using the bounded real lemma(BRL) and linear matrix inequalities (LMIs). Necessary and sufficient conditions for the existence of an H^∞ controller of any order were given in terms of three LMIs.

For the H^∞ control problem, most of papers exist on linear nondelayed systems, but very few papers exist on linear delay systems. Since time-delay is frequently a source of instability and encountered in various engineering systems such as chemical process, hydraulic, and rolling mill systems, etc., the stability problems of time-delay systems have received considerable attentions over the decades. Because systems often include some disturbances and time-delays, it is necessary to study the H^∞ control problem for time-delay systems. Recently, Lee *et al.*[4] and Choi *et al.*[5] extended the state feedback H^∞ controller design method proposed by Petersen[6] to state delayed systems and both state and input delayed systems, respectively. But when all states of a linear time-delay system are not available, these methods cannot be applied.

Therefore this paper presents an output feedback H^∞ controller design method for linear systems with delayed states and inputs. The Lyapunov function is used to develop the robust stability for time-delays. And a sufficient condition for BRL of time-delay systems is presented. This BRL is analogous to the BRL of nondelayed systems. We present the sufficient conditions for the existence of an H^∞ controller using LMIs and briefly explain how to construct such controllers from the positive definite solutions of their LMIs. Finally, we give a simple example to illustrate the validity of the proposed design procedure.

II. Problem Formulation

Consider the delay system described by the state-space equations of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t)+A_{d_1}x(t-d_1)+B_1w(t)+B_2u(t)+B_{d_2}u(t-d_2) \\ z(t) &= C_1x(t)+D_{11}w(t)+D_{12}u(t) \\ y(t) &= C_2x(t)+D_{21}w(t) \\ x(t) &= 0,\quad t<0,\quad x(0)=x_0 \end{aligned} \tag{1}$$

where $x(t)\in\mathbf{R}^n$ is the state, $w(t)\in\mathbf{R}^l$ is the square-integrable disturbance input vector, $u(t)\in\mathbf{R}^m$ is the control, $z(t)\in\mathbf{R}^p$ is the controlled output, $y(t)\in\mathbf{R}^q$ is the measurement output, d_1 and d_2 are positive real numbers, and A, Ad_1, B_1, B_2, Bd_2, C_1, C_2, D_{11}, D_{12}, and D_{21} are constant matrices with appropriate dimensions. Consider the kth order linear time-invariant dynamic controller

$$\begin{aligned} \dot{\hat{x}}(t) &= A_K\hat{x}(t)+B_Ky(t) \\ u(t) &= C_K\hat{x}(t)+D_Ky(t) \end{aligned} \tag{2}$$

where $\hat{x}(t)\in\mathbf{R}^k$ is the controller state. When we apply the

Manuscript received May 2, 1997; accepted September 29, 1997.

E. T. Jeung is with Department of Control and Instrumentation Engineering, Changwon National University.

D. C. Oh is with Department of Electronic Information, Konyang University.

H. B. Park is with School of Electronic and Electrical Engineering, Kyungpook National University.

control (2) to the delay system (1), the closed-loop system from w to z is given by

$$\begin{aligned}\dot{\xi}(t) &= A_{cl}\xi(t) + A_{cl1}\xi(t-d_1) + \\ &\quad A_{cl2}\xi(t-d_2) + B_{cl}w(t) + B_{cl2}w(t-d_2) \\ z(t) &= C_{cl}\xi(t) + D_{cl}w(t)\end{aligned} \tag{3}$$

where

$$\begin{aligned}\xi(t) &= \begin{bmatrix} x(t) \\ \hat{x}(t)\end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A+B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K\end{bmatrix}, \\ A_{cl1} &= \begin{bmatrix} A_{d_1} & 0 \\ 0 & 0_k\end{bmatrix}, \quad A_{cl2} = \begin{bmatrix} B_{d_2}D_KC_2 & B_{d_2}C_K \\ 0 & 0_k\end{bmatrix}, \\ B_{cl} &= \begin{bmatrix} B_1+B_2D_KD_{21} \\ B_KD_{21}\end{bmatrix}, \quad B_{cl2} = \begin{bmatrix} B_{d_2}D_KD_{21} \\ 0_{k\times l}\end{bmatrix}, \\ C_{cl} &= [C_1+D_{12}D_KC_2 \quad D_{12}C_K], \\ D_{cl} &= D_{11}+D_{12}D_KD_{21}.\end{aligned} \tag{4}$$

Here, we gather all controller parameters into the single variable

$$K = \begin{bmatrix} D_K & C_K \\ B_K & A_K\end{bmatrix} \tag{5}$$

and introduce the shorthands:

$$\begin{aligned}A_{00} &= \begin{bmatrix} A & 0 \\ 0 & 0_k\end{bmatrix}, \quad A_{10} = \begin{bmatrix} A_{d_1} \\ 0_{k\times n}\end{bmatrix}, \\ B_{00} &= \begin{bmatrix} B_2 & 0 \\ 0 & I_k\end{bmatrix}, \quad B_{10} = \begin{bmatrix} B_1 \\ 0_{k\times l}\end{bmatrix}, \\ B_{20} &= \begin{bmatrix} B_{d_2} \\ 0_{k\times m}\end{bmatrix}, \quad C_{00} = \begin{bmatrix} C_2 & 0 \\ 0 & I_k\end{bmatrix}, \\ C_{10} &= [C_1 \quad 0_{p\times k}], \quad D_{10} = [D_{12} \quad 0_{p\times k}], \\ D_{20} &= \begin{bmatrix} D_{21} \\ 0_{k\times l}\end{bmatrix}, \quad E_{10} = [I_n \quad 0_{n\times k}], \\ E_{20} &= [I_m \quad 0_{m \text{times} k}],\end{aligned} \tag{6}$$

then

$$\begin{aligned}A_{cl} &= A_{00}+B_{00}KC_{00}, \quad A_{cl1} = A_{10}E_{10}, \\ A_{cl2} &= B_{20}E_{20}KC_{00}, \quad B_{cl} = B_{10}+B_{00}KD_{20}, \\ B_{cl2} &= B_{20}E_{20}KD_{20}, \quad C_{cl} = C_{10}+D_{10}KC_{00}, \\ D_{cl} &= D_{11}+D_{10}KD_{20}.\end{aligned} \tag{7}$$

Note that (6) involves only plant data and that all matrices of (7) are affine form of the controller data K. We consider the design of a stabilizing controller data K which yields the closed-loop system with H^∞ norm bounded above by a specified number. To help our results, we need to review a well-known result.

Lemma 1: For any symmetric matrix $L=\begin{bmatrix} L_{11} & L_{12} \\ L_{12}^T & L_{22}\end{bmatrix}$, the following statements are equivalent.

i) $L < 0$

ii) $L_{11} < 0, \quad L_{22}-L_{12}^TL_{11}^{-1}L_{12} < 0$

iii) $L_{22} < 0, \quad L_{11}-L_{12}L_{22}^{-1}L_{12}^T < 0.$ □

III. Sufficient Conditions of Stability and H^∞ Norm Bound for Delay Systems

In this section, we discuss the stability condition of the system (3) and present a sufficient condition which stabilizes the closed-loop system (3) and guarantees the H^∞ norm bound.

Lemma 2: Consider the system (3) and suppose that the disturbance input is zero for all time. If there exist positive definite matrices P, R_1, and R_2 such that

$$\begin{aligned}&A_{cl}^TP+PA_{cl}+E_{10}^TR_1E_{10}+PA_{10}R_1^{-1}A_{10}^TP+C_{00}^TK^TE_{20}^TR_2E_{20}KC_{00} \\ &+PB_{20}R_2^{-1}B_{20}^TP < 0,\end{aligned} \tag{8}$$

then (3) is asymptotically stable for all $d_1, d_2 \geq 0$.

Proof: Let's define a Lyapunov functional $V(\xi, t)$ as follows:

$$\begin{aligned}V(\xi, t) = &\ \xi^T(t)P\xi(t)+\int_{t-d_1}^{t}\xi^T(\tau)E_{10}^TR_1E_{10}\xi(\tau)d\tau \\ &+\int_{t-d_2}^{t}\xi^T(\tau)C_{00}^TK^TE_{20}^TR_2E_{20}KC_{00}\xi(\tau)d\tau,\end{aligned} \tag{9}$$

then the corresponding Lyapunov derivative is given by

$$\frac{dV(\xi,t)}{dt} = \begin{bmatrix}\xi(t) \\ E_{10}\xi(t-d_1) \\ E_{20}KC_{00}\xi(t-d_2)\end{bmatrix}^T \begin{bmatrix} S & PA_{10} & PB_{20} \\ A_{10}^TP & -R_1 & 0 \\ B_{20}^TP & 0 & -R_2\end{bmatrix} \begin{bmatrix}\xi(t) \\ E_{10}\xi(t-d_1) \\ E_{20}KC_{00}\xi(t-d_2)\end{bmatrix} \tag{10}$$

where

$$S = A_{cl}^TP+PA_{cl}+E_{10}^TR_1E_{10}+C_{00}^TK^TE_{20}^TR_2E_{20}KC_{00}. \tag{11}$$

Due to lemma 1, the matrix in (10) is negative definite if there exist positive definite matrices P, R_1, and R_2 satisfying (8). Therefore the Lyapunov derivative is always less than zero and (3) is asymptotically stable for all $d_1, d_2 \geq 0$. □

Lemma 3: Consider the system (3). Suppose that $\sigma_{\max}(D_{cl}) < \gamma$ and that there exist positive definite matrices P, R_1, and R_2 such that

$$\begin{aligned}&A_{cl}^TP+PA_{cl}+PA_{10}R_1^{-1}A_{10}^TP+E_{10}^TR_1E_{10}+\gamma^{-2}C_{cl}^TC_{cl} \\ &+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})^T \\ &+PB_{cl2}(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}B_{cl2}^TP+PB_{20}R^{-1}B_{20}^TP \\ &+[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T] \\ &\times K^TE_{20}^TRE_{20}K[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T]^T < 0\end{aligned} \tag{12}$$

where

$$R = [R_2^{-1}-E_{20}KD_{20}(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^TK^TE_{20}^T]^{-1} > 0. \tag{13}$$

Then the following statements are true:

i) the system (3) is asymptotically stable for all $d_1, d_2 \geq 0$.

ii) $\|T_{zw}(j\omega)\|_\infty \leq \gamma$ where

$$T_{zw}(j\omega)= D_{cl}+C_{cl}(j\omega I-A_{cl}-A_{cl1}e^{-j\omega d_1}-A_{cl2}e^{-j\omega d_2})^{-1} \times(B_{cl}+B_{cl2}e^{-j\omega d_2}). \quad (14)$$

Proof: Suppose that $\sigma_{\max}(D_{cl})<\gamma$ and that there exist positive definite matrices P, R_1, and R_2 satisfying (12) and (13).

i) Using lemma 1, we can obtain an equivalent expression of (12) as follows:

$$\begin{bmatrix} \bar{S}+\gamma^{-2}C_{cl}^TC_{cl} & C_{00}^TK^TE_{20}^T & \gamma^{-2}C_{cl}^TD_{cl}+PB_{cl} \\ E_{20}KC_{00} & -R_2^{-1} & E_{20}KD_{20} \\ \gamma^{-2}D_{cl}^TC_{cl}+B_{cl}^TP & D_{20}^TK^TE_{20}^T & -(I-\gamma^{-2}D_{cl}^TD_{cl}) \end{bmatrix}<0 \quad (15)$$

where

$$\bar{S}=A_{cl}^TP+PA_{cl}+PA_{10}R_1^{-1}A_{10}^TP+E_{10}^TR_1E_{10}+PB_{20}R_2^{-1}B_{20}^TP. \quad (16)$$

By premultiplying and postmultiplying $\begin{bmatrix} I&0&0\\0&0&I\\0&I&0\end{bmatrix}$ and $\begin{bmatrix} I&0&0\\0&0&I\\0&I&0\end{bmatrix}$ respectively, to (15), we get

$$\begin{bmatrix} \bar{S}+\gamma^{-2}C_{cl}^TC_{cl} & \gamma^{-2}C_{cl}^TD_{cl}+PB_{cl} & C_{00}^TK^TE_{20}^T \\ \gamma^{-2}D_{cl}^TC_{cl}+B_{cl}^TP & -(I-\gamma^{-2}D_{cl}^TD_{cl}) & D_{20}^TK^TE_{20}^T \\ E_{20}KC_{00} & E_{20}KD_{20} & -R_2^{-1} \end{bmatrix}<0. \quad (17)$$

It implies that

$$\bar{S}+C_{00}^TK^TE_{20}^TR_2E_{20}KC_{00}+\gamma^{-2}C_{cl}^TC_{cl}<0 \quad (18)$$

and also implies (8).

ii) Let's define a positive definite matrix Q:=(negative of left hand side of (12)). Adding and subtracting

$$j\omega P+A_{cl1}^TPe^{j\omega d_1}+PA_{cl1}e^{-j\omega d_1}+A_{cl2}^TPe^{j\omega d_2}+PA_{cl2}e^{-j\omega d_2},$$

then

$$\begin{aligned}
&(-j\omega I-A_{cl}^T-A_{cl1}^Te^{j\omega d_1}-A_{cl2}^Te^{j\omega d_2})P \\
&+P(j\omega I-A_{cl}-A_{cl1}e^{-j\omega d_1}-A_{cl2}e^{-j\omega d_2}) \\
&-(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})^T \\
&-PB_{cl2}(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}B_{cl2}^TP-PB_{20}R^{-1}B_{20}^TP \\
&+A_{cl2}^TPe^{j\omega d_2}+PA_{cl2}e^{-j\omega d_2} \\
&-\{C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T\}K^TE_{20}^T \\
&\times RE_{20}K\{C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T\}^T \\
&=\gamma^{-2}C_{cl}^TC_{cl}+Q+W_1(j\omega)
\end{aligned} \quad (19)$$

where

$$W_1(j\omega):=[PA_{10}e^{-j\omega d_1}-E_{10}^TR_1]R_1^{-1}[A_{10}^TPe^{j\omega d_1}-R_1E_{10}]. \quad (20)$$

Note that $W_1(j\omega)$ is positive semidefinite for all $\omega\in\mathbb{R}$. We define $\Phi(j\omega):=(j\omega I-A_{cl}-A_{cl1}e^{-j\omega d_1}-A_{cl2}e^{-j\omega d_2})^{-1}$ and $\hat{B}(j\omega):=B_{cl}+B_{cl2}e^{-j\omega d_2}$ for simplicity. By premultiplying and postmultiplying $\hat{B}^*(j\omega)\Phi^*(j\omega)$ and $\Phi(j\omega)\hat{B}(j\omega)$, respectively, to (19), we get

$$\begin{aligned}
&\gamma^{-2}T_{zw}^*(j\omega)T_{zw}(j\omega)-I+\hat{B}^*(j\omega)\Phi^*(j\omega)[Q+W_1(j\omega)]\Phi(j\omega)\hat{B}(j\omega) \\
&=-(I-\gamma^{-2}D_{cl}^TD_{cl}) \\
&+(\gamma^{-2}D_{cl}^TC_{cl}+B_{cl}^TP)\Phi(j\omega)\hat{B}(j\omega) \\
&+\hat{B}^*(j\omega)\Phi^*(j\omega)(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl}) \\
&+B_{cl2}^TP\Phi(j\omega)\hat{B}(j\omega)e^{j\omega d_2}+\hat{B}^*(j\omega)\Phi^*(j\omega)PB_{cl2}e^{-j\omega d_2} \\
&+\hat{B}^*(j\omega)\Phi^*(j\omega)A_{cl2}^TP\Phi(j\omega)\hat{B}(j\omega)e^{j\omega d_2} \\
&+\hat{B}^*(j\omega)\Phi(j\omega)PA_{cl2}\Phi(j\omega)\hat{B}(j\omega)e^{-j\omega d_2} \\
&-\hat{B}^*(j\omega)\Phi^*(j\omega)(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl}) \\
&\quad\times(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}(\gamma^{-2}D_{cl}^TC_{cl}+B_{cl}^TP)\Phi(j\omega)\hat{B}(j\omega) \\
&-\hat{B}^*(j\omega)\Phi^*(j\omega)PB_{cl2}(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}B_{cl2}^TP\Phi(j\omega)\hat{B}(j\omega) \\
&-\hat{B}^*(j\omega)\Phi^*(j\omega)[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl}) \\
&\quad\times(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T]K^TE_{20}^TRE_{20}K \\
&\quad\times[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T]^T\Phi(j\omega)\hat{B}(j\omega) \\
&-\hat{B}^*(j\omega)\Phi^*(j\omega)PB_{20}R^{-1}B_{20}^TP\Phi(j\omega)\hat{B}(j\omega)
\end{aligned} \quad (21)$$

for all $\omega\in\mathbb{R}$. Using the inequality

$$A^TB+B^TA\le A^TRA+B^TR^{-1}B \quad (22)$$

for any matrices A, B, $R>0$ with appropriate dimension, we can get the inequality as follows:

$$\begin{aligned}
&-\hat{B}^*(j\omega)\Phi^*(j\omega)[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl}) \\
&\quad\times(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T]K^TE_{20}^TRE_{20}K \\
&\quad\times[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T]^T\Phi(j\omega)\hat{B}(j\omega) \\
&-\hat{B}^*(j\omega)\Phi^*(j\omega)PB_{20}R^{-1}B_{20}^TP\Phi(j\omega)\hat{B}(j\omega) \\
&\le-\hat{B}^*(j\omega)\Phi^*(j\omega)[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl})(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T] \\
&\quad\times K^TE_{20}^TB_{20}^TP\Phi(j\omega)\hat{B}(j\omega)e^{j\omega d_2} \\
&-\hat{B}^*(j\omega)\Phi^*(j\omega)PB_{20}E_{20}K\times[C_{00}^T+(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl}) \\
&\quad(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1}D_{20}^T]^T\Phi(j\omega)\hat{B}(j\omega)e^{-j\omega d_2}
\end{aligned} \quad (23)$$

for all $\omega\in\mathbb{R}$. Using the inequality (23), (21) becomes

$$\begin{aligned}
&\gamma^{-2}T_{zw}^*(j\omega)T_{zw}(j\omega)-I \\
&\le-\hat{B}^*(j\omega)\Phi^*(j\omega)[Q+W_1(j\omega)]\Phi(j\omega)\hat{B}(j\omega) \\
&-[\hat{B}^*(j\omega)\Phi^*(j\omega)(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl}+PB_{cl2}e^{-j\omega d_2}) \\
&-(I-\gamma^{-2}D_{cl}^TD_{cl})]\times(I-\gamma^{-2}D_{cl}^TD_{cl})^{-1} \\
&\times[\hat{B}^*(j\omega)\Phi^*(j\omega)(\gamma^{-2}C_{cl}^TD_{cl}+PB_{cl}+PB_{cl2}e^{-j\omega d_2}) \\
&\quad-(I-\gamma^{-2}D_{cl}^TD_{cl})]^*
\end{aligned} \quad (24)$$

for all $\omega\in\mathbb{R}$. The right hand side of (24) is negative semidefinite for all $\omega\in\mathbb{R}$. Therefore $\|T_{zw}\|_\infty\le\gamma$. □

The Riccati inequality (12) in lemma 3 is similar to the Riccati inequality of BRL for nondelayed systems except terms related time-delays. That is, lemma 3 presents a sufficient condition that the time-delay system is stable and the H^∞ norm of the time-delay system is less than or equal to given $r>0$.

IV. Existence and Construction of H^{∞} Controllers

By applying the result of lemma 3 developed in the previous section, we present sufficient conditions for the existence of an H^{∞} controller for linear delay systems and also briefly explain how to construct such controllers from the positive definite solutions of three LMIs.

Using lemma 1, necessary and sufficient condition satisfying $R_1>0$, $R_2>0$, $\sigma_{\max}(D_{cl}) < \gamma$, (12), and (13) is

$$\begin{bmatrix} A_{cl}^TP+PA_{cl} & C_{00}^TK^TE_{20}^T & PB_{cl} & C_{cl}^T & PA_{10} & PB_{20} & E_{10}^T \\ E_{20}KC_{00} & -R_2^{-1} & E_{20}KD_2 & 0 & 0 & 0 & 0 \\ B_{cl}^TP & D_2^TK^TE_{20}^T & -\gamma I & D_{cl}^T & 0 & 0 & 0 \\ C_{cl} & 0 & D_{cl} & -\gamma I & 0 & 0 & 0 \\ A_{10}^TP & 0 & 0 & 0 & -R_1 & 0 & 0 \\ B_{20}^TP & 0 & 0 & 0 & 0 & -R_2 & 0 \\ E_{10} & 0 & 0 & 0 & 0 & 0 & -R_1^{-1} \end{bmatrix} < 0. \tag{25}$$

Equivalently, this condition with the notation of (7) can be represented as

$$\Sigma+\Lambda\Pi K\Theta^T+\Theta K^T\Pi^T\Lambda < 0 \tag{26}$$

where

$$\begin{aligned} \Lambda &= diag\{P,\ I,\ I,\ I,\ I,\ I,\ I\}, \\ \Pi &= [B_{00}^T\ \ E_{20}^T\ \ 0\ \ D_{10}^T\ \ 0\ \ 0\ \ 0]^T, \\ \Theta &= [C_{00}\ \ 0\ \ D_{20}\ \ 0\ \ 0\ \ 0\ \ 0]^T, \end{aligned} \tag{27}$$

and

$$\Sigma=\begin{bmatrix} A_{00}^TP+PA_{00} & 0 & PB_{10} & C_{10}^T & PA_{10} & PB_{20} & E_{10}^T \\ 0 & -R_2^{-1} & 0 & 0 & 0 & 0 & 0 \\ B_{10}^TP & 0 & -\gamma I & D_{11}^T & 0 & 0 & 0 \\ C_{10} & 0 & D_{11} & -\gamma I & 0 & 0 & 0 \\ A_{10}^TP & 0 & 0 & 0 & -R_1 & 0 & 0 \\ B_{20}^TP & 0 & 0 & 0 & 0 & -R_2 & 0 \\ E_{10} & 0 & 0 & 0 & 0 & 0 & -R_1^{-1} \end{bmatrix}. \tag{28}$$

Lemma 4: Consider the problem of finding some matrix K satisfying (26). (26) is solvable for some K if and only if

$$\Pi_{\perp}^T\Lambda^{-1}\Sigma\Lambda^{-1}\Pi_{\perp} < 0, \tag{29}$$

$$\Theta_{\perp}^T\Sigma\Theta_{\perp} < 0, \tag{30}$$

where $\Pi_{\perp}$ and $\Theta_{\perp}$ are orthogonal complements of Π and Θ, respectively.

Proof: See [2], [3], and [7]. □

Using the conditions (29) and (30) in lemma 4, we can eliminate the controller data K to obtain conditions including only P. To simplify the condition (29) and (30), we partition P and P^{-1} as

$$P=\begin{bmatrix} Y & N \\ N^T & ? \end{bmatrix},\quad P^{-1}=\begin{bmatrix} X & M \\ M^T & ? \end{bmatrix} \tag{31}$$

where $X,\ Y\in \mathbf{R}^{n\times n}$, $M,\ N\in \mathbf{R}^{n\times k}$, ? means irrelevant. And we can choose $\begin{bmatrix} I & 0 & -B_2 & 0 \\ 0 & 0 & -D_{12} & I \end{bmatrix}^T$ and $[W_1^T\ \ W_2^T]^T$ which are orthogonal complements of $[B_{00}^T\ E_{20}^T\ D_{10}^T]^T$ and $[C_2\ D_{21}]^T$, respectively, then

$$\Pi_{\perp}=\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -B_2^T & -D_{12}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},\quad \Theta_{\perp}=\begin{bmatrix} W_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ W_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}. \tag{32}$$

Inequalities (29) and (30) are simplified to

$$\tilde{\Pi}^T\tilde{X}\tilde{\Pi} < 0, \tag{33}$$

$$\tilde{\Theta}^T\tilde{Y}\tilde{\Theta} < 0, \tag{34}$$

respectively, where

$$\tilde{\Pi}=\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ -B_2^T & -D_{12}^T & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},\quad \tilde{\Theta}=\begin{bmatrix} W_1 & 0 & 0 & 0 & 0 \\ W_2 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$\tilde{X}=\begin{bmatrix} XA^T+AX & 0 & XC_1^T & B_1 & A_{d_1} & B_{d_2} & X \\ 0 & -R_2^{-1} & 0 & 0 & 0 & 0 & 0 \\ C_1X & 0 & -\gamma I & D_{11} & 0 & 0 & 0 \\ B_1^T & 0 & D_{11}^T & -\gamma I & 0 & 0 & 0 \\ A_{d_1}^T & 0 & 0 & 0 & -R_1 & 0 & 0 \\ B_{d_2}^T & 0 & 0 & 0 & 0 & -R_2 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & -R_1^{-1} \end{bmatrix},$$

$$\tilde{Y}=\begin{bmatrix} A^TY+YA & YB_1 & C_1^T & YA_{d_1} & YB_{d_2} & I \\ B_1^TY & -\gamma I & D_{11}^T & 0 & 0 & 0 \\ C_1 & D_{11} & -\gamma I & 0 & 0 & 0 \\ A_{d_1}^TY & 0 & 0 & -R_1 & 0 & 0 \\ B_{d_2}^TY & 0 & 0 & 0 & -R_2 & 0 \\ I & 0 & 0 & 0 & 0 & -R_1^{-1} \end{bmatrix}.$$

Theorem 1: Consider the system (1) and let $[W_1^T\ \ W_2^T]^T$ be orthogonal complement of $[C_2\ \ D_{21}]^T$. If there exist positive definite matrices R_1, R_2, X, and Y satisfying the LMIs (33), (34), and

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \tag{35}$$

then γ-suboptimal H^{∞} control problem is solvable. Moreover, if

$$Rank(I-XY) = k < n \tag{36}$$

for some $X>0$, $Y>0$ satisfying (33)-(35), then there exist γ-suboptimal H^{∞} controllers of order k.

Proof: There exists a positive definite matrix P satisfying (31) if and only if the inequality $X-Y^{-1}\geq 0$ holds. This inequality is equivalent to (35). The rest of the proof is mentioned before. □

Note that theorem 1 does not present the computation of the controller itself, but existence conditions of H^∞ controllers. To compute H^∞ controllers, first compute some solutions (X, Y) satisfying LMIs (33)-(35), second compute two full-column-rank matrices $M,\ N \in \mathbf{R}^{n\times k}$ such that

$$MN^T = I - XY. \tag{37}$$

Then the unique solution P is obtained from the linear equation:

$$\begin{bmatrix} Y & I \\ N^T & 0 \end{bmatrix} = P \begin{bmatrix} I & X \\ 0 & M^T \end{bmatrix}. \tag{38}$$

Note that (38) is always solvable when $Y>0$ and M has full-column-rank[8]. Given P, since (26) is an LMI in K, γ-suboptimal H^∞ controllers can be computed as any solution K of (26). Note that the order of the controller is determined by the dimension of P.

V. Example

Consider a system of (1) with

$$A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix},\ A_{d_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_{d_2} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix},$$
$$C_1 = [1\ 1],\ C_2 = [1\ 3],\ D_{11} = 0,\ D_{12} = 1,\ D_{21} = 1.$$

Let $\gamma = 2$, $R_1 = I_2$, and $R_2 = 0.1$, then one pair of positive definite solutions satisfying (33)-(35) is

$$X = \begin{bmatrix} 1.5097 & 0.1178 \\ 0.1178 & 0.5944 \end{bmatrix},\quad Y = \begin{bmatrix} 2.9154 & 1.4445 \\ 1.4445 & 2.9718 \end{bmatrix},$$

and one pair of solutions satisfying (37) is

$$M = \begin{bmatrix} -0.9478 & -0.3190 \\ -0.3190 & 0.9478 \end{bmatrix},\quad N = \begin{bmatrix} 3.7683 & 0 \\ 2.6973 & -0.0804 \end{bmatrix}.$$

The positive definite solution of (38) is

$$P = \begin{bmatrix} 2.9154 & 1.4445 & 3.7683 & 0 \\ 1.4445 & 2.9718 & 2.6973 & -0.0804 \\ 3.7683 & 2.6973 & 6.3458 & -0.0242 \\ 0 & -0.0804 & -0.0242 & 0.0423 \end{bmatrix}$$

and one of the H^∞ controllers satisfying (26) is

$$K = \left[\begin{array}{c|cc} -1.4482 & -2.0728 & -0.1789 \\ \hline -0.4018 & -2.3945 & 0.0774 \\ 5.9769 & 10.9707 & -7.0006 \end{array}\right]$$

VI. Conclusion

In this paper, we have developed a sufficient condition of the bounded real lemma for linear systems with delayed states and inputs, and we have proposed the output feedback H^∞ controller design method. The proposed BRL is extension of the BRL without any delayed terms. Existence conditions of an output feedback H^∞ controller for linear delay systems are given in terms of three LMIs. We briefly explained how to construct such controllers from the positive definite solutions of their LMIs. The output feedback H^∞ controller guarantees not only the asymptotic stability of the closed loop system but also the H^∞ norm bound.

Acknowledgement

This work was supported by Korea Science and Engineering Foundation under Grant 961-0924-148-1.

References

[1] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and H^∞ control problems," *IEEE Trans. Automat. Contr.*, Vol. 34, No. 8, pp. 831-847, 1989.

[2] P. Gahinet and P. Apkarian, "An LMI-based parameterization of all H^∞ controllers with applications," in *Proc. IEEE Conf. Dec. Contr.*, pp. 656-661, 1993.

[3] T. Iwasaki and R. E. Skelton, "All controllers for the general H^∞ control problem: LMI existence conditions and state space formulas," *Automatica*, Vol. 30, No. 8, pp. 1307-1317, 1994.

[4] J. H. Lee, S. W. Kim, and W. H. Kwon, "Memoryless H^∞ controllers for state delayed systems," *IEEE Trans. Automat. Contr.*, Vol. 39, No. 1, pp. 159-162, 1994.

[5] H. H. Choi and M. J. Chung, "Memoryless H_∞ controller design for linear systems with delayed state and control," *Automatica*, Vol. 31, No. 6, pp. 917-919, 1995.

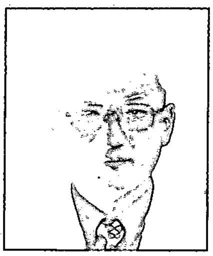
[6] I. R. Petersen, "Disturbance attenuation and H^∞ optimization: A design method based on the algebraic Riccati equation," *IEEE Trans. Automat. Contr.*, Vol. 32, No. 5, pp. 427-429, 1987.

[7] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, 1994.

[8] A. Packard, K. Zhou, P. Pandey, and G. Becker, "A collection of robust control problems leading to LMI's," in *Proc. IEEE Conf. Dec. Contr.*, pp. 1245-1250, 1991.

[9] M. S. Mahmoud and N. F. Al-Muthairi, "Design of robust controllers for time-delay systems," *IEEE Trans. Automat. Contr.*, Vol. 39, No. 5, pp. 995-999, 1994.

[10] J. C. Shen, B. S. Chen, and F. C. Kung, "Memoryless stabilization of uncertain dynamic delay systems: Riccati equation approach," *IEEE Trans. Automat. Contr.*, Vol. 36, No. 5, pp. 638-640, 1991.



Eun-Tae Jeung was born in Taegu, Korea, in 1966. He received the B. S., M. S., and Ph. D. degrees in electronic engineering from Kyungpook National University, Taegu, Korea, in 1991, 1993, and 1996, respectively. He is currently a Full-time Lecturer in the Department of Control and Instrumentation Engineering at Changwon National University. His current research interests include robust control, H^∞ control, time delay system, and fuzzy control. He is a member of IEEE, KITE, ICASE, and KFIS.

Hong-Bae Park, for a photograph and biography, see p. 38 of the June 1996(vol. 1, no. 2) issue of this Journal.

Do-Chang Oh received B. S., M. S., andPh. D. degrees in electronic engineering from Kyungpook National University, Taegu, Korea, in 1991, 1993, and 1997, respectively. He is currently a Full-time Lecturer in the Department of Electronic Information at Konyang University, Nonsan, Chungnam. His current research interests include robust control, H^∞ control, model and controller reduction, and time delay control. He is a member of KITE and ICASE.