A Sediment Concentration Distribution Based on a Revised Prandtl's Mixing Theory

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ABSTRACT: Prandtl's mixing length theory was modified to obtain a power velocity distribution in which the coefficient and the exponent are variable over a range from :

$$\bar{u} / \sqrt{\tau_o/\rho} = 4/\kappa y^{1/4}$$

$$\bar{u} / \sqrt{\tau_o/\rho} = 7/\kappa y^{1/7}$$

A simple suspended-sediment concentration distribution was developed from the modified velocity distribution :

$$C/C_a = (a/y)^{w/(\beta \kappa \sqrt{\tau_o/\rho})}$$

With nominal values of β=1.0, κ=0.4 and visual accumulation tube values of the fall velocity, the comparison between the theory and field measurements by the USGS on the Rio Grande is fair. Doubling the value of the exponent results in a good comparison. Further researches are needed for choosing the values of β, κ, and fall velocity values, and consideration on the effects of large-scale turbulence and secondary flows are necessary for them. In a pragmatic sense, on any gaging sites the close analysis of very detailed measurements can establish its specific coefficient and exponent.

1. Prandtl's Mixing Length Theory and Logarithmic Velocity Distribution.

Prandtl's notion of turbulence was, in essence, that small masses of fluid are exchanged without losing much, if any, of their momentum (velocity). When mass from a higher-velocity region passes a point, the instantaneous velocity at the point becomes greater than the temporal mean velocity; when mass is from a lower-velocity region the instantaneous velocity is less than the

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mean velocity($u=\bar{u}+u'$ and $u=\bar{u}-u'$, respectively). Prandtl's (1926) original mixing length theory and the logarithmic velocity distribution begin with the equation for shear:

$$\tau = -\rho\,\overline{u'v'} \tag{1}$$

in which ρ is the mass density of the fluid and the bar over the product denotes mean value.

The concepts for relating the turbulent fluctuation u' and v' to the mean velocity gradient, $d\bar{u}/dy$ are:

$$u' \propto l\,\frac{d\bar{u}}{dy} \tag{2}$$

$$v' \propto u' \tag{3}$$

$$l = \kappa y \tag{4}$$

$$\frac{\tau}{\rho} = \kappa^2 y^2 \left(\frac{d\bar{u}}{dy}\right)^2 \tag{5}$$

where y is the distance from the boundary, l is the mixing length (the effective distance which the mass of the fluid off density, ρ, moves laterally in the turbulent flow), and κ absorbs all the proportionality coefficients.

Prandtl then assumed the internal shear was equal to the boundary shear:

$$\tau = \tau_o \tag{6}$$

The differential equation becomes:

$$\frac{d\bar{u}}{\sqrt{\tau_o/\rho}} = \frac{1}{\kappa}\frac{dy}{y} \tag{7}$$

which can be integrated as:

$$\frac{\bar{u}}{\sqrt{\tau_o/\rho}} = \frac{1}{\kappa}\ln y + \text{constant} \tag{8}$$

which agrees very well with Nikuradse's measurements in pipes.

2. Rouse's Concentration Distribution

Rouse (1937) used the logarithmic velocity distribution, O'Brien's (1933) diffusion equation for suspended sediment, mixing coefficients for momentum (velocity) and suspended sediment, and assumed that the two mixing coefficients were proportional.

$$Cw = -\in_s \frac{dc}{dy} \tag{9}$$

$$\tau = \rho \in_m \frac{d\bar{u}}{dy} \tag{10}$$

$$\tau = (1 - \frac{y}{D})\, \tau_o \tag{11}$$

$$\in_s = \beta \in_m \tag{12}$$

Combining and integrating, one gets:

$$\frac{C}{C_a} = \left(\frac{D-y}{y} \cdot \frac{a}{D-a}\right)^{\frac{w}{\beta\kappa\sqrt{\tau_o/\rho}}} \tag{13}$$

where the a is the value of y where $C = C_a$ and,

C = sediment concentration at level y

C_a = reference sediment concentration at level a

D = flow depth

w = fall velocity

y = height above the channel bed

β = coefficient of proportionality between $\in_s$ and $\in_m$ · β=1.0 unless otherwise noted.

κ = the "universal" mixing length coefficient (not necessarily prandtl's κ, and probably variable, not constant)

τ_o = tractive force at stream bed

ρ = density of water

$\in_s$ = mixing coefficient for sediment

$\in_m$ = mixing coefficient for momentum

If a β value of about 1.5 is used, Rouse's equation agrees quite well with measurements. However, it is difficult to explain why sediment is mixed more effectively than momentum. Rouse's internal shear varies linearly in the vertical (as it should), whereas Prandtl's internal shear is constant. There are other details of Prandtl's concept that should also be examined further. However, the best reason to revisit Prandtl and Rouse is the difficulty in integrating the products of their equations to obtain the suspended sediment load.

$$q_s = \int_o^D \bar{u} c\, dy \tag{14}$$

3. Laursen's (1980) Revision of Prandtl's Assumptions

Several of Prandtl's statements or assumptions do not stand up well to critical scrutiny: (1) the instantaneous turbulence components are not proportional to the product of the mixing length and the mean velocity gradient, (2) the temporal averaged turbulent shear is not proportional to some measure of the product of the two temporal averaged turbulence components, and (3) the internal shear is not a constant equal to the boundary shear.

A few simple revisions are sufficient to make the derivation of the logarithmic velocity distribution more rigorous. Let

$$\sqrt{\overline{u'^2}} \propto l \frac{d\bar{u}}{dy} \tag{15}$$

$$\sqrt{\overline{u'^2}} \propto \sqrt{\overline{\nu'^2}} \tag{16}$$

$$\frac{\overline{u'\nu'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{\nu'^2}}} = R \tag{17}$$

where R is a correlation coefficient equal to:

$$R = (1-\frac{y}{D}) \tag{18}$$

This assumption(Laursen 1980) will result in the familiar logarithmic velocity distribution when the correct internal shear distribution

$$\tau = (1-\frac{y}{D})\,\tau_o$$

and Prandtl's mixing length assumption

$$\tau = \kappa\, y$$

are used.

4. Laursen's Concentration Distribution (Laursen 1980)

If, now, the difference between the mixing coefficients for sediment and momentum is partly due to the correlation coefficient R, the relationship between the two is:

$$\in_s = \frac{\beta \in_m}{(1-y/D)} \tag{19}$$

and the mixing coefficient for sediment becomes:

$$\epsilon_s = \beta\, \kappa\, \sqrt{\tau_o / \rho}\, y \tag{20}$$

The differential form of the sediment diffusion equation is then:

$$\frac{dC}{C} = -\frac{\mathrm{w}dy}{\beta\, \kappa\, \sqrt{\tau_o / \rho}\, y} \tag{21}$$

which integrates simply to:

$$\frac{C}{C_a} = \left(\frac{a}{y}\right)^{\frac{\mathrm{w}}{\beta\kappa\sqrt{\tau_o/\rho}}} \tag{22}$$

Note that β in this equation is not the same as the β value in the Rouse equation. Both β values are found by forcing the theoretical equations to fit measured data. A comparison of the two equations is shown in Fig. 1.

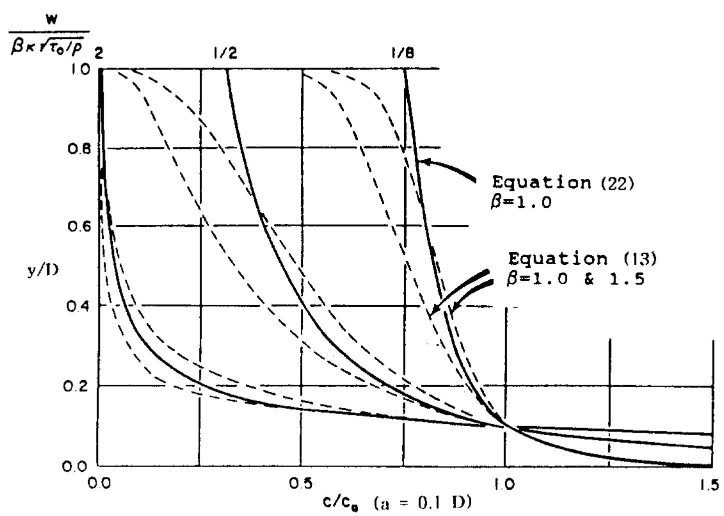

Fig. 1. Comparison of Rouse (Eq. 13) and Laursen (Eq. 22) concentration distributions. β=1.0, κ=0.4, a=0.1D (Laursen 1980)

5. Jung's (1989) First Revision of Laursen's Revision of Prandtl's Assumptions

Laursen's mathematically simpler sediment concentration distribution equation still requires integration involved in computation to find the suspended sediment load. Use of a power distribution for the velocity would make the integration much easier. However, there would be a logical inconsistency since the logarithmic velocity distribution was used in deriving the concentration distribution. The first attempt to resolve this inconsistency was a change in the assumption for the mixing length to:

$$l = \kappa y^{\gamma} \tag{23}$$

Using Laursen's other assumptions(Eqs. 15, 16, 17, 18), combining terms, and integrating results in a power velocity distribution.

$$\bar{u} = (\frac{1}{\kappa}) \sqrt{\frac{\tau_o}{\rho}} \frac{1}{(-\gamma+1)} y^{-\gamma+1} \tag{24}$$

If γ is taken as 3/4, $(-\gamma + 1)$ is 1/4; if γ is 6/7, $(-\gamma + 1)$ is 1/7. These are the typical values found in the literature of rough and smooth boundaries, and

$$\bar{u} = (\frac{4}{\kappa}) \sqrt{\tau_o/\rho} y^{1/4} \tag{25}$$

$$\bar{u} = (\frac{7}{\kappa}) \sqrt{\tau_o/\rho} y^{1/7} \tag{26}$$

Note that κ values in this power equation do not have to be the same as the κ value in the logarithmic distribution equation (usually taken as about 0.4 or less). A comparison of the power equation (Eq. 26) and the logarithmic equation with the universal equation indicates that the power distribution with 1/7 exponent fits well with logarithmic velocity distribution, as shown in Fig. 2.

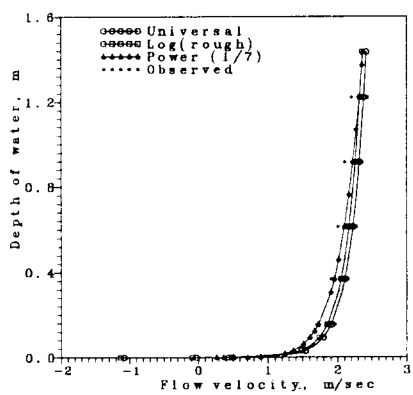

Fig. 2. Comparison of velocity distributions

Power-law formulas were used before logarithmic equations were used (Daily and Harleman 1966). The general power velocity distribution equation (Izbash and Khaldre 1971) is:

$$\frac{\overline{u}}{U_{max}} = \left(\frac{y}{D}\right)^x \tag{27}$$

where the terms are as follows:

$U_{\max}$ = velocity at the water surface

u = time-averaged flow velocity at a distance y from the bed

x = a power index, usually between 1/4 to 1/7

The velocity $\overline{u}$ at level y is given by Equation(27), and since the mean velocity $\overline{U}$ times D is equal to the discharge per unit width, q is:

$$q = \overline{U} D = \int_o^D \overline{u}\, dy \tag{28}$$

$$q = \overline{U} D = \int_o^D U_{\max} \left(\frac{y}{D}\right)^x dy \tag{29}$$

which results in:

$$q = \overline{U} D = \left(\frac{1}{x+1}\right) U_{\max} D \tag{30}$$

So,

$$U_{\max} = (x+1)\, \overline{U} \tag{31}$$

Therefore,

$$\frac{\overline{u}}{(x+1)\, \overline{U}} = \left(\frac{y}{D}\right)^x \tag{32}$$

$$\frac{\overline{u}}{\overline{U}} = (x+1)\left(\frac{y}{D}\right)^x \tag{33}$$

Continuing to follow Laursen's assumption, the associated concentration distribution equation can be obtained. The mixing coefficients, the differential diffusion equation, and the final formulation are:

$$\in_m = \sqrt{\tau_o/\rho}\ (1-y/D)\ \kappa y^{\gamma} \tag{34}$$

$$\in_s = \frac{\beta \in_m}{(1-\frac{y}{D})} = \beta \sqrt{\tau_o/\rho}\ \kappa y^{\gamma} \tag{35}$$

$$\frac{dC}{C} = \frac{w}{\in_s} dy = \frac{w}{\beta\kappa \sqrt{\tau_o/\rho}} \frac{dy}{y^{\gamma}} = -\frac{z}{y^{\gamma}} dy \tag{36}$$

where

$$z = \frac{w}{\beta\kappa \sqrt{\tau_o/\rho}} \tag{37}$$

and

$$\frac{C}{C_a} = e^{\frac{z}{-\gamma+1} a^{-\gamma+1}} / e^{\frac{z}{-\gamma+1} y^{-\gamma+1}} \tag{38}$$

If $\gamma = 3/4$ (rough boundary):

$$\frac{C}{C_a} = e^{4za^{1/4}} / e^{4zy^{1/4}} = e^{4z(a^{1/4} - y^{1/4})} \tag{39}$$

If $\gamma = 6/7$ (smooth boundary) :

$$\frac{C}{C_a} = e^{7za^{1/7}} / e^{4zy^{1/7}} = e^{7z(a^{1/7} - y^{1/7})} \tag{40}$$

6. Jung's (1993) Second Revision of Laursen's Revision of Prandtl's Assumptions

Unfortunately the first attempt to find a simpler pair of equations resulted in a complex concentration distribution equation to go with the simpler power velocity distribution. In the second attempt at simplicity, the assumption $R=(1-y/D)$ for the correlation coefficient was modified to:

$$R = (1-\frac{y}{D}) y^{2\gamma-2} \tag{41}$$

Retaining the previous assumption for mixing length, the velocity distribution becomes:

$$\bar{u} = \frac{1}{\kappa}\sqrt{\frac{\tau_o}{\rho}}\ \frac{1}{-2\gamma+2}\ y^{-2\gamma+2} \tag{42}$$

which appears to be slightly different form Equation (24), but when γ is 7/8 or 13/14 the exponent is 1/4 or 1/7 and results in Equations (25) and (26). The mixing coefficients, differential diffusion equation, and final concentration distribution equation become:

$$\in_m = \sqrt{\tau_o/\rho}\ (1-\frac{y}{D})\ \kappa y^{2\gamma-1} \tag{43}$$

$$\in_s = \beta\kappa\sqrt{\tau_o/\rho y}$$

$$\frac{dC}{C} = -\frac{w}{\beta\kappa\sqrt{\tau_o/\rho}}\ \frac{dy}{y} = -z\ \frac{dy}{y} \tag{44}$$

$$\frac{C}{C_a} = (\frac{a}{y})^{\frac{w}{\beta\kappa\sqrt{\tau_o/\rho}}} \tag{45}$$

Equation (45) is the same as Laursen's concentration distribution, Equation (22). The three concentration distribution equations (Rouse, Eq. 13; Laursen, Eq. 22; and Jung, Eq. 40) are compared in Fig. 3.

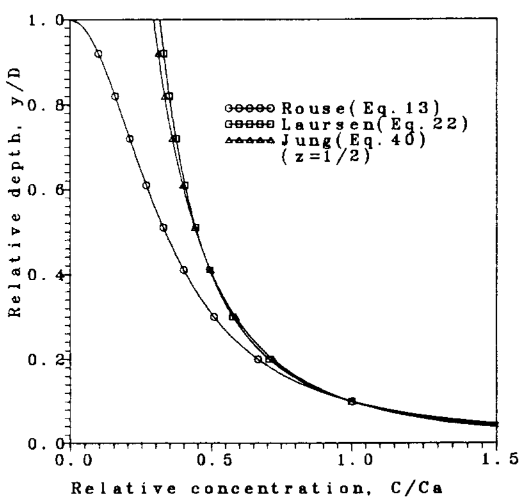

Fig. 3. Comparison of Equations of (13), (22), and (40). β=1.0, κ=0.4, a=0.1D

7. Comparison of Laursen's Concentration Distribution with Rio Grande Measurements

Fig. 4 shows a comparison between the theoretical concentration curves using equation (45) and two sets of measurements made by the U. S. Geological Survey (USGS) at gaging stations 2243 and 2249 on the Rio Grande (Culbertson et al. 1972). The bed material and suspended sediment samples were divided into four fractions in the USGS data (Fig. 5), and these were used for the computations. The z-values were based on nominal values $\kappa=0.4$, $\beta=1.0$, visual accumulation tube fall velocity, and measured values of slope and depth.

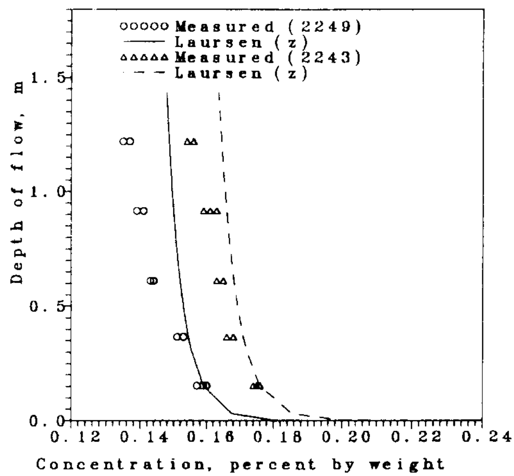

Fig. 4. Comparison of the concentration distribution (Eq. 45) with measure-ments at San Marcial, Rio Grande River

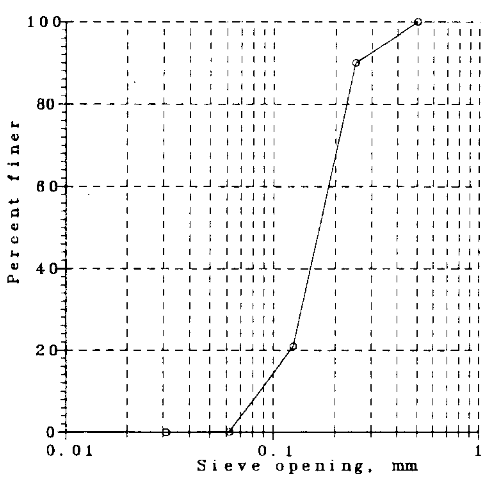

Fig. 5. Typical bed material size distribution at San Marcial, Rio Grande River, Station 2249

The reference concentration used in Fig. 4 was at the lowest sampling point. If the mid-depth sampling point had been used, the theoretical curve would seem to go through the measured data much better. However, while this better fit is more apparent than real, the better fit does lead to a relevant question: where should the sediment sample be taken? For the fine sediment the reference level should approach mid-depth; for the coarse sediment it should approach the bed, for the mixed sediment it is somewhere in between and depends partly on the sediment problem to be evaluated.

In Fig. 6, the same data are used as in Fig. 4, and the value of z is arbitrarily doubled. The Laursen curves now go through the measured points as well as one could expect. Note that the lowest sampling elevation is still the reference level. The question, then, is why the "correct" z value might be twice the nominal z value. The z value would be larger if the value was smaller. The value could be in the order of 0.32; this would increase z by a factor of 1.25 (0.4/0.32). The diameter determined by the visual accumulation(VA) tube is presumed to be the diameter of a sphere falling by itself, and the sediment particles are presumed to have the same fall velocity (or time of fall) as measured by the VA tube. This should give the "correct" fall velocity if the concentration of the particles in the tube is the same as the concentration in the flow, and if the turbulence in the tube is the same as the turbulence in the stream. If the concentration in the tube is greater than in the stream, a factor larger than unity should be applied. The tube's turbulence is caused by the falling particles and can probably be safely neglected. Not enough has been done to permit estimating the effect of the stream turbulence on the fall velocity of suspended sediment particles. One can argue that since non-spherical particles tend to fall such that the drag forces are at the maximum, the turbulence of a scale which would rotate particles would result in a larger fall velocity. If these two velocity effects need multiplying factors of 1.20 and 1.33, respectively, the three factors together would increase he nminal value of z by a

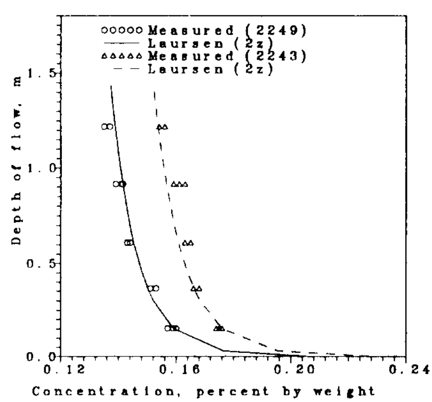

Fig. 6. Comparison of concentration distributions (Eq. 45) using corrected z with measures at San Marcial, Rio Grande River

factor of two (1.25*1.2*1.33). A β value of less than unity would have a similar effect, but there is no evidence that this would be a reasonable supposition.

More researches to clarify these issues are needed, but it is not unreasonable to believe that the z used in Fig. 6 is just as valid as the z in Fig. 4, if not more so.

It is probably not reasonable to expect Prandtl's mixing length theory (or any variation of it) to be entirely satisfying in describing either the velocity distribution or the concentration distribution of a real river or, even, of a laboratory flume. There are other aspects of the flow that can have substantial effects: secondary currents that are quite steady and persistent, such as the spirals induced by bends, and vortices that are more random, close to the scale of the river depth, and which could also be considered very large-scale turbulence-but which are not the phenomenon described by Prandtl's turbulence.

The masses of fluid moving up and down lead to Prandtl's theory of mixing length. The large-scale eddies and vortices are different phenomena and are not approximated by Prandtl's theory. Secondary flows, such as vertical- and horizontal-axis vortices, can be very important in sediment problems. They could be more important in the turbulent mixing of sediment than in the turbulent mixing of velocity (momentum).

8. Average Sediment Concentration Based on the Simpler Concentration Distribution Equation and a Power Velocity Distribution.

From the general relationship among suspended lead, concentration, velocity, and depth,

$$q_s = \eta \int_{\alpha}^{D} \bar{u} C dy = \eta C_m \bar{U} D \tag{46}$$

where is a coefficient related to units of the variables, and

$$C_m = \int_{\alpha}^{D} \frac{\bar{u}}{\bar{U}} C \frac{dy}{D} \tag{47}$$

where is equal to $2d_{50}$, a value that may be changed in further research. From Equation (45) for the sediment concentration distribution,

$$C = C_a \left(\frac{a}{y}\right)^{\frac{w}{\beta\kappa\sqrt{\frac{\tau_o}{\rho}}}}$$

Finally from Equations (33), (45),and (47),

$$C_m = \int_{\alpha}^{D} \frac{\bar{u}}{\bar{U}} C \frac{dy}{D}$$

$$C_m = \int_{\alpha}^{D} (x+1) \left(\frac{y}{D}\right)^x \left(\frac{1}{D}\right) C_a \left(\frac{a}{y}\right)^z dy \tag{48}$$

$$C_m = (x+1) D^{-x-1} C_a a^z \left[\frac{1}{x - z + 1} y^{x-z+1}\right]_{\alpha}^{D} \tag{49}$$

or, the average sediment concentration in a vertical is

$$C_m = \frac{J C_a a_z}{(J-z) D^J} (D^{J-z} - \alpha^{J-z}) \tag{50}$$

where

$$J = x+1 = 5/4 \sim 8/7 \tag{51}$$

In Equation (50), is assumed to have a very small value near the bed of the river at a level equal to $2d_{50}$.

9. Comparison of Equations (40) and (45)

Computed values for the vertical concentration using Equation (40) and Equation (45), which is the same as Equation (22), are compared in Fig. 3. The two curves are quite consistent, but there are interesting differences. When computed values for a standard rectangular channel are considered, the C/C_a values using Equation (45) are always less than the C/C_a values using Equation (40), but the difference is negligible. Also, there is no big difference in computed values between 3/4 and 6/7 as the γ value, (or velocity powers of 1/4 and 1/7), as shown in Fig. 7. Generally, these two curves show good consistency. It is concluded, therefore, that computational ease can be an appropriate factor in choosing an approximate equation.

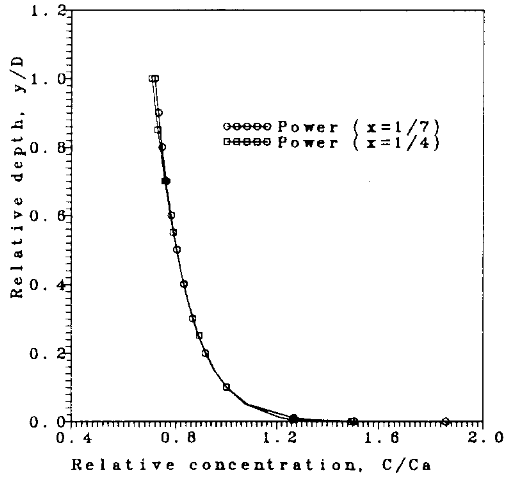

Fig. 7. Comparison of the concentration distribution using x=1/4 and x=1/7, β=1.0, κ=0.4, a=0.1D, z=0.1874

Also correlation coefficients (Equations (18) and (41)) and mixing lengths (Equations (4) and (23)) were examined graphically, as shown in Fig. s 8 and 9, respectively. Differences are apparent in the correlation coefficients in the lower level of flow, while the two mixing lengths do not agree above the mid-point of the flow. However, the values of these two quantities, L and R, cannot be directly measured, and it is not possible to say which is a better approximation. Both give velocity and concentration distributions which are acceptable approximations.

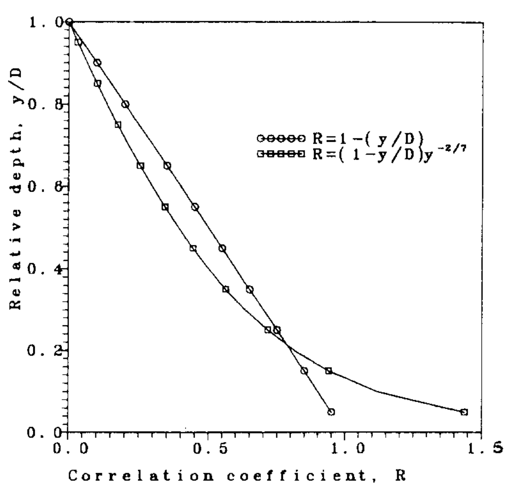

Fig. 8. Comparison of correlation coefficients. γ=6/7

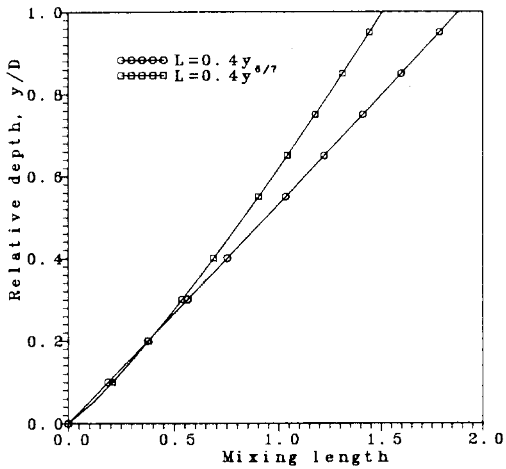

Fig. 9. Comparison of mixing lengths. κ=0.4

10. Discussion and Conclusions

This study demonstrates that simple forms of the velocity and concentration distribution equations are sufficiently accurate. The most uncertain step in the computational procedure presented is related to the value of the Rouse number $z=w/(\kappa\beta\sqrt{\tau_o/\rho})$. Variables in this parameter are not yet completely understood. The fall velocity value needed in the computation procedure is the fall velocity of a natural sediment particle in the turbulent flow and in the presence of many other particles. The is for the flow in a natural channel of complex platform and with complex (and variable) roughness elements. Note that the value of is likely to be different, based on different assumptions of the turbulence structure and relation to the mean velocity gradient. The factor is the difference between the mixings of momentum and sediment in suspension above the correlation coefficient used in the derivations presented in this study. The scant evidence cited here suggests that the restricted β have a value of about one, and that any variation be small. Other papers will examine: (1) the ability to predict suspended sediment loads by the integration of these proposed equations, and (2) possible errors to be guarded against in a single-point sampling.

The velocity distribution and the concentration distribution can be written with parameters containing variables generally accepted. Better agreement with measures can be obtained by using coefficients and exponents derived from measurements at the specific gaging site. The difference between nominal "theoretical" values of the requisite parameters and variables and the empirical values based on measurements can sometimes, and to some extent, be explained by known variations in such things as the and the fall velocity. Obviously the mixing length theory has a simplified concept which does not fully describe the flow behavior and mixing by large-scale secondary and tertiary flow phenomena which are something between the mean flow behavior and random turbulent eddies.

Further research on the characteristics and effects of secondary components of the mean flow and of transient vortices, which might be classified as the large-scale turbulence, would be interesting and challenging. However, each gaging site would be unique at least in some way, and it is very doubtful that the behaviors of these flow features could ever be well predicted for specific sites. Field measurements of velocity and concentration distributions should be sufficient to permit adjustment of the coefficients and exponents of the proposed equations.

Prandtl's basic concepts are still useful, but his concepts can be revised slightly and refined in order to better describe fluid flow phenomena. The refinements offered herein result in a better, but not perfect, description of the sediment concentration distribution in the vertical. With more and better measurements, other assumptions of the mixing length and the correlation coefficient might be made to give better results. There is always temptation to say it would be wise to wait for better measurements and understanding of all the secondary factors involved, such as the fall velocity of particles in a turbulent field and in the presence of other particles of various sizes and the effects of secondary flow in a supposedly two-dimensional flow. However, measurements in establishing a gaging site, and occasional measurements subsequently, as is done today, should be good enough for acceptable determination of suspended sediment loads by single-point sampling.

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