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ON IRREDUCIBLE 3-MANIFOLDS

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ABSTRACT. This paper deals with certain conditions under which irreducibility of a 3-manifold is preserved under attaching a 2-handle along a simple closed curve (and then, if necessary, capping off a 2-sphere boundary component by a 3-ball).

1. Introduction

Let $\mathbb{M}$ be a 3-manifold and γ a simple closed curve in $\partial\mathbb{M}$. Let $\mathbb{N}(\gamma)$ be a regular neighborhood of γ in $\partial\mathbb{M}$ and endow the 3-ball $\mathbb{B}^3$ with the product stucture $\mathbb{D}^2 \times \mathbb{I}$. If $\phi : \partial\mathbb{D}^2 \times \mathbb{I} \to \mathbb{N}(\gamma)$ is a homeomorphism, we define a new manifold $\mathbb{M}_\gamma$ to be $\mathbb{M}$ with a 2-handle attached along γ: that is,

$$\mathbb{M}_\gamma = \mathbb{M} \cup_\phi \mathbb{B}^3$$

If $\{\gamma_i\}_{i=1}^n$ is a finite collection of pairwise disjoint simple closed curves on $\partial\mathbb{M}$, then $\mathbb{M}_{\gamma_1,\cdots,\gamma_n}$ is defined to be $(\cdots((\mathbb{M}_{\gamma_1})_{\gamma_2})\cdots)_{\gamma_n}$. The homeomorphism type does not depend on the ordering of the γ_i. Hence, we often denote $\mathbb{M}_{\gamma_1,\cdots,\gamma_n}$ by $\mathbb{M}_{\{\gamma_1\}_{i=1}^n}$. $\mathbb{M}_\gamma^+$ denotes the manifold obtained from $\mathbb{M}_\gamma$ by capping off each 2-sphere boundary component of $\mathbb{M}_\gamma$ with a 3-ball. Kneser [5] proved that every closed orientable 3-manifold, different from S^3, can be built up as a finite connected sum of prime 3-manifolds, and according to Milnor [6] this decomposition is unique up to order and homeomorphism. With the exception of a 3-manifold homeomorphic to $S^3, S^1 \times S^2$, a 3-manifold is prime if and only if it is irreducible. Thus, the classification problem for compact 3-manifold is reduced to that for irreducible 3-manifolds. Assuming a 3-manifold to be irreducible avoids certain technical difficulties as well as the Poincaré Conjecture. In this

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paper, it will be proved that under certain conditions, irreducibility is preserved under attaching a 2-handle along a simple closed curve (and then, if necessary, capping off a 2-sphere boundary component by a 3-ball).

2. Definitions and Remarks

This chapter deals with some definitions and remarks related to our main topic. Some geometric concepts are also introduced.

DEFINITION 2.1 [1]. Let $\mathbb{M}$ be a 3-manifold and F a connected surface which is either properly embedded in $\mathbb{M}$ or contained in $\partial\mathbb{M}$. We say that F is *compressible* if one of the followings is satisfied:

(1) F is a 2-sphere which bounds a 3-ball in $\mathbb{M}$,
(2) F is a 2-disk and either $F \subset \partial\mathbb{M}$ or there is a 3-ball $B \subset \mathbb{M}$ with

$$\partial\mathbb{B} \subset F \cup \partial\mathbb{M},$$

(3) there is a 2-disks $\mathbb{D} \subset \mathbb{M}$ such that $\mathbb{D} \cap F = \partial\mathbb{D}$ and $\partial\mathbb{D}$ is not contractible in F.

The 2-disk $\mathbb{D}$ in (3) is called a compressing disk for F in $\mathbb{M}$. We say that F is incompressible if it is not compressible. If F is not connected, we say that F is incompressible in the case that all components of F are incompressible.

DEFINITION 2.2 [7]. Let $\gamma_1, \cdots, \gamma_n$ be pairwise disjoint 2-sided simple closed curves in the boundary of a 3-manifold $\mathbb{M}$. γ in $\partial\mathbb{M}$ is said to be *coplanar with* $\{\gamma_i\}_{i=1}^n$ if γ bounds a disk in $\partial\mathbb{M}_{\gamma_1,\cdots,\gamma_n}$.

REMARK [7]:. Let γ and γ' be disjoint simple closed curves in the boundary of a 3-manifold $\mathbb{M}$. If $\mathbb{M}$ is compact and γ is coplanar with γ', then precisely one of the following possibilities occures:

(1) γ bounds a disk in $\partial\mathbb{M} - \gamma'$,
(2) γ is parallel in $\partial\mathbb{M}$ to γ',
(3) γ separates $\partial\mathbb{M}$ into two components, one of which is a punctured torus $\mathbb{T}$ and $\gamma' \subset \mathbb{T}$ does not separate $\mathbb{T}$.

DEFINITION 2.3 [7]. Let $\mathbb{M}$ be a 3-manifold and P a disk with holes properly embedded in $\mathbb{M}$, and $\gamma \subset \partial\mathbb{M}$ a 2-sided simple closed curve. P is said to be a *pre-sphere* with respect to γ in $\mathbb{M}$ if

(1) $\partial P \subset \partial\mathbb{M} - \gamma$,
(2) each component of ∂P is coplanar with γ in $\partial\mathbb{M}$, and
(3) P^+ does not bound a 3-ball in M_γ^+, where P^+ is a natural extension of P to a 2-sphere in M_γ^+.

REMARK:. In condition (3) in Definition 2.3, let $D_0, \ldots, D_n$ be the disks in M_γ^+ attached to P along respective boundary components of P to produce P^+. If it were the case that some of the D_i are nested in a disk, say D_0, then we choose an innermost disk, D_1 say, among all of the disks D_i which are nested in D_0 and introduce a collar on $D_0, c_1 : D_0 \times I \to M_\gamma^+$, with $c_1(x, 0) = x$ such that $c_1(D_0 \times 1) \subset IntM_\gamma$ and $c_1(D_0 \times 1)$ cuts P^+ into two disks, one of which, denoted by D_1' contains D_1, and the other one contains all ∂D_i excepts ∂D_1. Then, push D_1' onto the disk D_1^* bounded by $\partial D_1'$ in $c_1(D_0 \times 1)$ by an isotopy across the 3-ball $c_1(D_0 \times 1)$. And then choose the innermost disk, D_2 say, among all $D_i \subset D_0$ except D_1 and introduce a collar on D_0 in $M_\gamma^+, c_2 : D_0 \times I \to M_\gamma^+$, with $c_2(x,0) = x$ such that $c_2(D_0 \times I) \subset IntM_\gamma, c_2(D_0 \times I)$ does not contain D_1^*, and $c_2(D_0 \times 1)$ cuts P^+ into two disks, one of which, denoted by D_2', contains D_2 and the other one contains all ∂D_i except ∂D_2. Then push D_2' onto the disk D_2^* bounded by $\partial D_2'$ in $c_2(D_0 \times 1)$ by an isotopy across the 3-ball $c_2(D_0 \times I)$. In the same way, we push the other disks D_i into $IntM_\gamma$ sequentially so that the more inner in D_0 a disk, the deeper into $IntM_\gamma$ it is pushed.. Keeping this process until all the D_i are embedded in $IntM_\gamma$, we can assume that P^+ is embedded. (2) P^+ is unique up to isotopy in M_γ^+.

3. Irreducibility

Even though the following observation was made by Jaco [3], we prove it here with hopes of enhancing the understanding of the geometry appearing in our main topic.

LEMMA 3.1. *Let M be a 3-manifold with ∂M compressible and γ a simple closed curve in $\partial M - \gamma$ incompressible. Let δ be a simple closed* *curve in $\partial M - \gamma$ which is not contractible in M. If δ is coplanar with γ, then $\partial M - \delta$ is incompressible in M*

PROOF. If δ is parallel in ∂M to γ, then the result is trivial. Hence, we assume δ is not parallel in ∂M to γ. Since δ is not contractible in $\partial M - \gamma$, δ must separate ∂M into two components, one of which is a once-punctured torus T with γ a non-separating curve in T. Let $\partial M = S \cup_\delta T$ for a surface S. Suppose $\partial M - \delta$ is compressible in M. If S is compressible in M, there is a properly embedded disk D_1 in M such that $\partial D_1 \subset S$ and ∂D_1 does not bound a disk in S. Since $\partial M - \gamma = S \cup_\delta (T - \gamma)$, ∂D_1 does not bound a disk in $\partial M - \gamma$. Hence, D_1 is a compressing disk for $\partial M - \gamma$ in M, which leads to a contradiction. Thus, S is incompressible in M, and so T must be compressible in M. Let D be a compressing disk for T in M. If ∂D is parallel to δ in T, there is an annulus A in T with $\partial A = \partial D \cup \delta$, and so $D' = A \cup_{\partial D} D$ is a 2-disk with $\partial D' = \delta$. If ∂D is not freely homotopic to δ in T, ∂D is non-separating in T. Hence, D compresses T to a disk. Now, let $N(D)$ denote a regular neighborhood of D with a product structure $D \times [-1, 1]$ in M. Then, $D' = (T - \partial D \times (-1,1)) \cup (D \times \{-1, 1\})$ is a 2-disk and $\partial D' = \delta$. However, this is absurd because δ is not contractible in M. □

The following fact was asserted, without proof, by Przytycki [7]. Our proof is a modification of Jaco's argument used in proving his Handle Attaching Theorem [3]. However, it will turn out that the translation of Jaco's argument into this context requres considerable effort. This is, to the best of our knowledge, the first time the proof of this potentially valuable result has appeared in detail.

THEOREM 3.1. *Let M be a 3-manifold with ∂M compressible and γ a 2-sided simple closed curve with $\partial M - \gamma$ incompressible in M. If M is irreducible, then so is M_γ^+.*

PROOF. Suppose that M_γ^+ is not irreducible. Then, there exists an embedded 2-sphere S in M_γ^+ such that S does not bound a 3-ball in M_γ^+. since S can be made disjoint from 3-handles by general position, we assume $S \subset M_\gamma$. Moreover, by general position, S meets the 2-handle attached along γ in parallel disks. Since M is irreducible, S cannot be embedded in M and so $S \cap \partial M \neq \phi$. Since S is compact and S meets the 2-handle in disks, $S \cap \partial M$ is a finite collection of simple closed curves parallel to γ in ∂M. Let $P = S \cap M$. Then, P is a properly embedded disk with holes in M such that

(1) $\partial P \subset \partial M - \gamma$,
(2) each component of ∂P is coplanar with γ in ∂M, and
(3) $P^+ = S$ in M_γ^+.

That is, there is a pre-sphere with respect to γ in M. Hence, to prove the theorem, it suffices to show that there is no pre-sphere in M with respect to γ. Let $\mathcal{P}$ be the collection of properly embedded disks with holes in M so that $P \in \mathcal{P}$ if and only if there exists a simple closed curve $\gamma \subset \partial M$ such that $\partial M - \gamma$ is incompressible in M and P is a pre-sphere with respect to γ in M. We will show that $\mathcal{P}$ is empty, thus establishing the theorem. Suppose that $\mathcal{P} \neq \phi$. Choose an element $P \in \mathcal{P}$ so that the Euler characteristic $\chi(P)$ is maximal. Then, there exists a simple closed curve $\gamma \subset \partial M$ such that $\partial M - \gamma$ is incompressible in M and P is a pre-sphere with respect to γ. Let $\mathbb{D}$ be the collection of compressing disks for ∂M in M. Since ∂M is compressible in M, the collection $\mathbb{D}$ is not empty. Each disk in $\mathbb{D}$ meets γ because $\partial M - \gamma$ is incompressible in M. If a simple closed curve $\delta \subset \partial M$ is coplanar with γ and is not contractible in M, then by lemma 3.1, $\partial M - \delta$ is incompressible in M. Since maximality of $\chi(P)$ implies that no component of ∂P is contractible in M, for any component δ of ∂P, $\partial M - \delta$ is incompressible in M. Hence, it follows that any disk in $\mathbb{D}$ must meet all components of ∂P and, moreover, P. Without loss of generality, we can assume that each disk is transverse to P. Choose a disk $D \in \mathbb{D}$ so that the number of components of intersection $D \cap P$ is minimal. We prove, by analyzing $D \cap P$, that this leads to the desired contradiction.

(1) Suppose that $D \cap P$ contains a simple closed curve. Then, choose a simple closed curve α from the components of $D \cap P$ which is innermost in D: that is, α bounds a disk Δ in D and $Int\Delta \cap P = \phi$. Futhermore, since P is planar, α separates P into two surfaces F_1 and F_2. If one, F_1 say, of the F_i is a disk, then there is a simple closed curve β which is a component of $D \cap P$ and innermost in F_1. That is, β bounds a disk Δ' in F_1 and $Int\Delta' \cap D = \phi$. Moreover, the curve β separates D into an annulus D_1 and a disk D_2 with $\partial D \subset D_1$. The disk $D_1 \cup \Delta'$ is a compressing disk for ∂M in M and has the property that after a small

isotopy, the number of components of the intersection $(D_1 \cup \Delta') \cap P$ is less than that of $D \cap P$. This contradicts the choice of D. Hence, we can assume neither F_1 nor F_2 is a disk. Let $P_i = F_i \cup \Delta$, and let F_i^+ denote the 2-disk obtained from F_i by attaching 2-disks along boundary components except α $i = 1, 2$ and desingularizing them as mentioned in the remark following definition 2.3. Then each P_i satisfies conditions (1) and (2) of definition 2.3. Observe that

(1) F_i^+ is a disk with $\partial F_i^+ = \alpha, \quad i = 1, 2,$
(2) $P_i^+ = F_i^+ \cup_\alpha \Delta, \quad i = 1, 2$ and
(3) $P^+ = F_1^+ \cup_\alpha F_2^+$.

Since P^+ does not bound a 3-ball in M_γ^+, from 2) and 3) in the above observation it follows that one of the 2-spheres P_i^+ does not bound a 3-ball in M_γ^+, and so P_i is a pre-sphere with respect to γ in M and is of Euler characteristic greater than $\chi(P)$. this contradicts the choice of P. Hence, we can conclude that no component of $D \cap P$ is a simple closed curve.

(2) Suppose that $D \cap P$ contains an inessential spanning arc in P. In other words, ther exists a component α of $D \cap P$ and an arc β in ∂P such that $\partial\alpha = \partial\beta$ and the curve $\alpha \cup \beta$ bounds disk Δ in P with $Int\Delta \cap D = \phi$. Now, we boundary-compress D along the disk Δ to get two properly embedded disks D_1 and D_2 with the property that the number of components of $D_i \cap P$ is less than that of $D \cap P$. Let δ_1 and δ_2 be arcs in ∂D such that $\partial D = \delta_1 \cup \delta_2, \partial\delta_1 = \partial\delta_2 = \partial\beta$, and $\partial D_i = \delta_i \cup \beta, i = 1, 2$. With appropriate orientations of δ_1, δ_2 and β, we have $\partial D_1 = \beta\delta_1$ and $\partial D_2 = \delta_2\beta^{-1}$ in $\pi_1(\partial M)$. If each ∂D_i is contractible in ∂M, then $\partial D = \delta_1\delta_2 = (\beta\delta_1)(\delta_2\beta^{-1}) = 1$ in $\pi_1(\partial M)$ and so ∂D is also contractible in ∂M. This contradicts the fact that D is a compressing disk for ∂M in M. Now, one of the D_i has non-contractible boundary of D. Hence, we can conclude that no component of $D \cap P$ is an inessential spanning arc in P.

(3) suppose $D \cap P$ contains an essential spanning arc in P. Let α be an essential arc which is a component of $D \cap P$ and outmost on D; that is, there is an arc $\beta \subset \partial D$ such that $\partial\alpha = \partial\beta$ and the imple closed curve $\alpha \cup \beta$ bounds a disk Δ in D with $Int\Delta \cap P = \phi$. At this point of time, we require some preparations to analyze case (3): To perform a boundary compression of P along Δ, it is necessary to prove the following lemma.

LEMMA 3.2. *β may be assumed disjoint from γ (by redefining γ to be parallel in ∂M to some component of ∂P, if necessary.)*

PROOF. Recall That M is irreducible, ∂M is compressible and $\partial M - \gamma$ is incompressible in M. Let S be the boundary component of ∂M containing γ. If γ is parallel in ∂M to a component, μ say, of ∂P, then there is an annulus A in ∂M with $\gamma, \mu \subset A$, along which γ can be isotoped to an appropriate side of μ, off β. Suppose no component of ∂P is parallel in ∂M to γ. Since no component of ∂P is contractible in M, every component of ∂P separates S into two components one of which is a once-punctured torus which contains γ, but is not separated by γ. Hence, there is a simple closed curve γ' in ∂M such that γ' separates S into two components W and a once-punctured torus T with $\gamma \subset T$ not separating, and all components of ∂P are parallel to γ' in ∂M. By lemma 3.1, $\partial M - \gamma'$ is incompressible in M and so W and T are. Note that W is not a disk. Hence, neither M_γ nor $M_{\gamma'}$ has 2-sphere boundary components; that is, $M_\gamma = M_\gamma^+$ and $M_{\gamma'} = M_{\gamma'}^+$. Since γ' bounds a disk in ∂M_γ, $(M_\gamma)_{\gamma'}^+$ is homeomorphic to M_γ. Thus, we get

$$M_\gamma = M_\gamma^+ \simeq M_{\gamma\gamma'}^+ = M_{\gamma'\gamma}^+.$$

Now, let $P_{\gamma'}^+$ be a natural extension of P in $M_{\gamma'}^+ = M_{\gamma'}$. Let ξ be an arbitrary component of ∂P. Let $D_\gamma(\text{resp.} D_{\gamma'})$ denote a 2-disk bounded by ξ in

$$P_\gamma^+ \subset M_\gamma = M_\gamma^+ (\text{resp.} P_{\gamma'}^+ \subset M_{\gamma'} = M_{\gamma'}^+).$$

Then, there is an embedded 3-ball $B^3 \subset Int M_{\gamma\gamma'}^+$ which meets D_γ and $D_{\gamma'}$ in complementary faces, and D_γ agrees with $D_{\gamma'}$ away from B^3. This implies that $P_{\gamma'}^+$ is (ambient) isotopic to P^+ in $M_{\gamma\gamma'}^+$ and so there is a homeomorphism of $M_{\gamma\gamma'}^+$ onto M_γ which carries $P_{\gamma'}^+$ to P^+. Since P^+ does not bound a 3-ball in M_γ^+, $P_{\gamma'}^+$ does not bound a 3-ball in $M_{\gamma'}^+$. Therefore, P is a pre-sphere in M with respect to γ'. Now, we see that $\partial M - \gamma'$ is incompressible, P is a pre-sphere in M with respect to γ', and every component of ∂P is parallel to γ' in ∂M. Therefore, by redefining γ to be γ', we can assume that some component μ of ∂P is parallel in

∂M to γ. Now, we can assume that γ is disjoint from β because γ can be isotoped off β along the appropriate annulus in ∂M containing γ and β. This establishes the lemma □

Now, we return to our proof.

CLAIM (1). It is not the case that both end points of α are contained in one component of ∂P.

Suppose that the component α of $D \cap P$ has both end points in a component δ of ∂P. Then a boundary compression of P at α along Δ results in two new disks with holes, P_1 and P_2. Since α is essential, $\chi(P) < \chi(P_i), i = 1, 2$. Let the end points of α separate δ into two arcs δ_1 and δ_2. Each simple closed curve $\delta_i \cup \beta \quad (i = 1, 2)$ is a new boundary component of P_i. Since β and γ are disjoint by lemma 3.2, we have $\partial P_i \subset \partial M - \gamma$. If one, $\delta_1 \cup \beta$ say, of the $\delta_i \cup \beta$ is not coplanar with γ, then P_1^+ is compressing disk for ∂M_γ^+ in M_γ^+. This contradicts Jaco's Handle Attaching Theorem. Hence, each P_i has a boundary whose components are all coplanar with γ. We want to show that at least one of the 2-sphere P_i^+ does not bound a 3-ball. Assume both P_i^+ bound 3-balls B_i^3, respectively. Let D_i be a disk bounded by $\delta_i \cup \beta$ in M_γ^+ to produce P_i^+. Let α separate the planar surface P into two surfaces F_1 and F_2 and F_i^+ denote the 2-disk obtained from F_i by attaching 2-disks in M_γ^+ along corresponding boundary components except $\alpha \cup \delta_i$ and desingularizing them. Then, we have $P_i = F_i \cup_\alpha \Delta$ and $P_i^+ = (F_i^+ \cup_\alpha \Delta) \cup_{\delta_i \cup \beta} D_i$. If one of the D_i is contained in the other one, $D_1 \subset D_2$ say, then let $D = cl(D_2 - D_1)$. We can assume that δ bounds the disk D in P^+. Now, we observe that $D \cup \Delta$ isotopes onto F_1^+ across the 3-ball B_1^3 and so P_2^+ isotopes onto P^+. This is, however, impossible because P^+ does not bound a 3-ball in M_γ^+. If it is not the case that one of the D_i is contained in the other one, then δ bounds the disk $D_1 \cup_\beta D_2$ in M_γ^+. Since $P_1 \cap P_2 = \Delta$ and $F_1^+ \cup D_1$ isotopes to Δ across B_1^3, P^+ isotopes to P_2^+. However, this is impossible because P^+ does not bound a 3-ball. Now, we see that one of the P_i is a pre-sphere in M with respect to γ, of Euler characteristic greater than $\chi(P)$. This contradicts the choice of P. So, we conclude that no component is a spanning arc with end points in one component of ∂P.

CLAIM (2). It is not the case that each end point of α is contained in a different component of ∂P.

Suppose that the component of $D \cap P$ is such an arc. This is, the end points of α lie in distinct components of ∂P, ξ_1 and ξ_2 say. Then a boundary compression of P along Δ results in a disk with holes, P', such that $\chi(P') > \chi(P)$ and $\partial P' = (\partial P - (\xi_1 \cup \xi_2)) \cup \xi$ where $\xi = \xi_1 \beta \xi_2 \beta^{-1}$, assuming ξ_1 and ξ_2 are appropriately oriented. We want to show that P' is a pre-sphere in M with respect to γ. Since β does not meet γ, $\xi \subset \partial M - \gamma$. Since ξ_1 and ξ_2 are contractible in ∂M_γ, ξ is contractible in ∂M_γ; that is, ξ is coplanar with γ in M. Let D_i be disks bounded by ξ_i in M_γ^+ with $D_i \subset P^+$. If each D_i is disjoint from $Int\beta$ (after a small isotopy if necessary),then ξ bounds the 2-disk $D_1 \cup_{l_1} (\beta \times [-1,1] \cup_{l_2} D_2)$ (where $l_1 \cup l_2 = \partial\beta \times [-1,1]$), and P'^+ isotopes onto P^+ across $\Delta \times [-1,1]$. Now, we assume $\beta \subset D_2$ without loss of generality. Since P is planar, $\xi_1 \subset D_2$. In this case, we push D_1 into $c(D_2 \times I) \subset IntM_\gamma^+$, where c is collar on D_0 in M_{γ^+} as described in the remark following definition 2.3. Performing boubdary-compressing along Δ, we observe that ξ bounds a disk, and P^+ isotopes onto P'^+ across the 3-ball, a boundary connected sum of the products $D_1 \times [0,1]$ and $\Delta \times [-1,1]$. Thus, P'^+ does not bound a 3-ball in M_γ^+, and so P' is a pre-sphere in M with respect to γ, with $\chi(P) < \chi(P')$. This contradicts the choice of P. Therefore, no component of $D \cap P$ can be a spanning arc which does not have both ends points in one component of ∂P.
By claim(1) and claim(2), no component of $D \cap P$ can be an essential spanning arc in P. Finally, it follows that $\mathcal{P}$ is empty. □

COROLLARY 3.1. *If γ is a simple closed curve in ∂H_k with $\partial H_k - \gamma$ incompressible, then $(H_k)_\gamma^+$ is irreducible.(Here, H_k denotes a handlebody of genus k.)*

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