

ON THE SPECTRAL PROPERTIES OF MULTIPLIERS

JONG-KWANG YOO

ABSTRACT. This note centers around the class $M(A)$ of multipliers on a Gelfand algebra A . This class is a large subalgebra of the Banach algebra $L(A)$. The aim of this note is to investigate some aspects concerning their local spectral properties of multipliers. In the last part of work we consider some applications to automatic continuity theory.

1. Introduction

In this paper we shall investigate some spectral properties of a multiplier defined on a complex commutative semi-simple Banach algebra A with unit. A commutative Banach algebra with unity element of norm one will be called a *Gelfand algebra*. In the following, let A be a Gelfand algebra, and let $M(A)$ denote the corresponding *multiplier algebra*, which consists of all mappings $T : A \rightarrow A$ with the property $(Tx)y = x(Ty)$ for all $x, y \in A$. Every multiplier T satisfies $T(xy) = (Tx)y$ for all $x, y \in A$. The concept of a multiplier was introduced by Helgason [7] as a bounded continuous function f defined on the maximal ideal space $\Delta(A)$ such that $f(\widehat{A}) \subseteq \widehat{A}$, where \widehat{A} denotes the Gelfand representation of the Banach algebra A . The most important example of a multiplier is obtained when we take $A := L_1(G)$, a group algebra of a locally compact abelian group G . In [13], Wendel states that the mapping $T : L_1(G) \rightarrow L_1(G)$ is a multiplier if and only if there exists a bounded regular complex Borel measure μ on G such that $T = \mu * f$ for all $f \in L_1(G)$, where the symbol $*$ denotes the usual convolution. For each $a \in A$, let $T_a : A \rightarrow A$ denote the corresponding

Received April 10, 1997. Revised July 2, 1997.

1991 Mathematics Subject Classification: Primary 47A11, Secondary 47B40.

Key words and phrases: Bishop's property, decomposition property, Gelfand algebra.

multiplication operator given by $T_a x = ax$ for all $x \in A$. By the semi-simplicity of A , we can identify A with the ideal $\{T_a : a \in A\}$ of $M(A)$. Let $\Delta(A)$ denote the *maximal ideal space* of A , the set of all nontrivial multiplicative linear functionals on A , and let $\hat{a} : \Delta(A) \rightarrow \mathbb{C}$ denote the Gelfand transform given by $\hat{a}(\phi) := \phi(a)$ for all $\phi \in \Delta(A)$. On $\Delta(A)$ we have to consider both the usual Gelfand topology, i.e. the relative *weak** topology, for which all the Gelfand transforms \hat{a} are continuous, and the hull-kernel topology, which is coarser than the Gelfand topology and for which some of the Gelfand transforms \hat{a} need not be hull-kernel continuous. Recall from ([3], Theorem 23.8) that these two topologies coincide if and only if the Banach algebra A is regular.

Given a complex Banach space X and the Banach algebra $L(X)$ of all bounded linear operators on X , an operator $T \in L(X)$ is said to have *Bishop's property* (β) if for every open subset U of the complex plane \mathbb{C} and for every sequence of analytic functions $f_n : U \rightarrow X$ such that $(T - \lambda)f_n(\lambda)$ converges uniformly to zero on each compact subset of U , it follows that $f_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$, uniformly on each compact subset of U . Obviously, the property (β) implies that T has the *single-valued extension property* which means that, for every open subset U of \mathbb{C} , the only analytic solution $f : U \rightarrow X$ of the equation $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$. An operator $T \in L(X)$ is said to have the *decomposition property* (δ) if given an arbitrary open covering $\{U, V\}$ of \mathbb{C} , every $x \in X$ has a decomposition $x = u + v$ where $u, v \in X$ satisfy $u = (T - \lambda)f(\lambda)$ on $\mathbb{C} \setminus \bar{U}$ and $v = (T - \lambda)g(\lambda)$ on $\mathbb{C} \setminus \bar{V}$ for some pair of X -valued analytic functions f and g on $\mathbb{C} \setminus \bar{U}$ and $\mathbb{C} \setminus \bar{V}$, respectively. It has been shown in [1] that the properties (β) and (δ) are dual to each other, i.e. an operator $T \in L(X)$ satisfies (β) if and only if the adjoint operator T^* satisfies (δ) and the corresponding statement remains valid if both properties are interchanged. Given an operator $T \in L(X)$ and a closed subset F of \mathbb{C} , let $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *analytic spectral subspace*, where $\sigma_T(x) \subseteq \mathbb{C}$ is the *local spectrum* of T at the point $x \in X$, i.e. the complement of the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in \mathbb{C} and an analytic function $f : U \rightarrow X$ such that $(T - \mu)f(\mu) = x$ for all $\mu \in U$. We say that a subspace Y of X is *T -divisible* if $(T - \lambda I)Y = Y$ for all $\lambda \in \mathbb{C}$. If $A \subseteq \mathbb{C}$ then the *maximal algebraic spectral subspace* $E_T(A)$

is the largest subspace of X on which all the restrictions of $T - \lambda I$, $\lambda \in \mathbb{C} \setminus A$, are surjective. It is obvious that $(T - \lambda I)E_T(F) = E_T(F)$ for all $\lambda \in \mathbb{C} \setminus F$ as well so that the class which we consider has a maximal element for the order by inclusion. In particular, $E_T(\phi)$ is the largest T -divisible subspace for the operator T .

2. Main results

A semi-simple Banach algebra A is called a *supremum norm algebra* if $\|x\| = \|\widehat{x}\|_\infty := \sup_{\phi \in \Delta(A)} |\widehat{x}(\phi)|$ for all $x \in A$.

PROPOSITION 1. *If A is a Gelfand algebra then $M(A)$ is a Gelfand subalgebra of $L(A)$. Moreover, if A is a supremum norm algebra, then so is $M(A)$ with its operator norm.*

PROOF. Clearly, every $T \in M(A)$ is linear. By the closed graph theorem, for any $T \in M(A)$, T is continuous and so $M(A) \subseteq L(A)$. Assume that there exists sequence $\{T_n\} \subseteq M(A)$ and $T \in L(A)$ such that $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$. Then for $x, y \in A$,

$$\|x(Ty) - (Tx)y\| \leq \|x(Ty) - x(T_n y)\| + \|(T_n x)y - (Tx)y\| \leq 2\|x\|\|y\|\|T_n - T\|,$$

and so $x(Ty) = (Tx)y$. Thus $M(A)$ is closed in $L(A)$, and hence $M(A)$ is a Gelfand subalgebra of $L(A)$. Assume that A is a supremum norm algebra, and let $T \in M(A)$. By theorem 3.5 of [12], there is a uniquely determined bounded Gelfand continuous function \widehat{T} on $\Delta(A)$ with the property that

$$\widehat{(Tx)} = \widehat{T}\widehat{x}, \quad \text{and} \quad \|\widehat{T}\|_\infty := \sup_{\phi \in \Delta(A)} |\widehat{T}(\phi)| \leq \|T\|$$

for all $x \in A$. Thus, we have

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|\widehat{x}\|_\infty=1} \|\widehat{(Tx)}\| \leq \sup_{\|\widehat{x}\|_\infty=1} \|\widehat{T}\|_\infty \|\widehat{x}\|_\infty = \|\widehat{T}\|_\infty.$$

Hence $\|T\| = \|\widehat{T}\|_\infty$, and so $M(A)$ is a supremum norm algebra. □

COROLLARY 2. *Assume that A is a semi-simple Gelfand algebra. Then $\widehat{T_a}(\phi) = \widehat{a}(\phi)$ for all $a \in A$ and $\phi \in \Delta(A)$.*

PROOF. $\widehat{a}(\phi)\widehat{x}(\phi) = \widehat{ax}(\phi) = \widehat{(T_ax)}(\phi) = \widehat{T_a}(\phi)\widehat{x}(\phi)$. □

We shall denote by $\sigma_p(T)$, $\sigma_{com}(T)$, $\sigma_r(T)$ the point spectrum, compression spectrum and residual spectrum of T , respectively.

COROLLARY 3. *Assume that A is a semi-simple Gelfand algebra. Then for any multiplication operator $T_a \in L(A)$,*

$$\sigma_p(T_a) \subseteq \sigma(a) \subseteq \sigma_p(T_a) \cup \sigma_r(T_a)$$

PROOF. By theorem 52.13 of [2] and corollary 2, $\sigma(a) = \widehat{a}(\Delta(A)) = \widehat{T_a}(\Delta(A))$. Let $\lambda \in \sigma_p(T_a)$. Then there exists a non-zero element $x \in A$ such that $(T_a - \lambda)x = 0$. Thus we have

$$(\widehat{a} - \lambda)\widehat{x} = \widehat{(ax - \lambda x)} = \widehat{[(T_a - \lambda)x]} = \widehat{0}$$

Since A is semi-simple, there exists an element $\phi_0 \in \Delta(A)$ such that $\widehat{x}(\phi_0) \neq 0$. It follows that $(\widehat{a} - \lambda)(\phi_0) = 0$, and so $\lambda = \widehat{a}(\phi_0) \in \sigma(a)$. Let T^* denote the topological dual of the operator T . Then

$$(T_a^* \phi)(x) = \phi(T_a x) = \widehat{(ax)}(\phi) = \widehat{a}(\phi)\widehat{x}(\phi) = \widehat{a}(\phi)\phi(x),$$

for all $\phi \in \Delta(A)$ and $x \in A$. Thus $T^* \phi = \widehat{a}(\phi) \cdot \phi$ and so $\widehat{a}(\phi) \in \sigma_p(T_a^*)$. The second inclusion follows from the well known inclusion ([2], p. 249)

$$\sigma_p(T_a^*) = \sigma_{com}(T_a) \subseteq \sigma_p(T_a) \cup \sigma_r(T_a)$$
 □

THEOREM 4. *If A is a C^* -algebra then $M(A)$ is a C^* -algebra.*

PROOF. Let $T^*x := (Tx^*)^*$ for all $x \in A$. Then $(T^*x)y = (y^*Tx^*)^* = ((Ty^*)x^*)^* = x(Ty^*)^* = xT^*y$ for all $x, y \in A$, and so $T^* \in M(A)$. It suffices to show that the mapping $T \rightarrow T^*$ is an involution. It is easily checked that $T^{**} = T$, $(TS)^* = S^*T^*$ and $(\lambda T + S)^* = \bar{\lambda}T^* + S^*$ for all $\lambda \in \mathbb{C}$ and $T, S \in M(A)$. It remains to be shown that $\|TT^*\| = \|T\|^2$. Let U denote the closed unit ball of A . Then U is self-adjoint. Thus we have

$$\|T^*\| = \sup_{x \in U} \|T^*x^*\| = \sup_{x \in U} \|(T^*x^*)^*\| = \sup_{x \in U} \|Tx\| = \|T\|.$$

Thus $\|TT^*\| \leq \|T\|^2$. If $x \in A$ then $\|x\| = \sup_{y \in U} \|yx\|$, by proposition 7.1.8 of [3]. Hence

$$\begin{aligned} \|TT^*\| &= \sup_{x \in U} \|TT^*x\| = \sup_{x \in U} \sup_{y \in U} \|TT^*xy\| \geq \sup_{x \in U} \|TT^*(xx^*)\| \\ &= \sup_{x \in U} \|(Tx)(T^*x^*)\| = \sup_{x \in U} \|Tx\|^2 = \|T\|^2. \end{aligned}$$

This completes the proof. □

A Banach algebra A endowed with an algebra involution $*$ is called a Banach $*$ -algebra. It follows from ([3], Theorem 36.2) that an involution is automatically continuous for a semi-simple Banach algebra A .

COROLLARY 5. Assume that A is a semi-simple Gelfand Banach $*$ -algebra. Then $M(A)$ is a semi-simple Gelfand Banach $*$ -algebra with respect to the involution given by $T^*(x) := T(x^*)^*$ for all $T \in M(A)$ and $x \in A$.

We say that the operator T has *finite ascent* if for every $\lambda \in \mathbb{C}$ there is a natural number $n \in \mathbb{N}$ such that $\text{Ker}(T - \lambda)^n = \text{Ker}(T - \lambda)^{n+1}$. It is well known that every hyponormal operator has finite ascent. In particular, generalized scalar operators fall in this class, see [10]. A Banach algebra A is called *semi-prime* in the sense that every two-sided ideal I of A with $I^2 = \{0\}$ is trivial.

PROPOSITION 6. *Assume that A is a semi-prime Gelfand algebra. Then every multiplier $T \in M(A)$ has finite ascent.*

PROOF. Let $T \in M(A)$ and $x \in A$ such that $T^2x = 0$. Since $(Tx)^2 = xT^2x = 0$ and $(Tx)a(Tx) = T(xaTx) = xaT^2x = 0$ for all $a \in A$, we conclude that the two-sided ideal I generated by Tx satisfies $I^2 = \{0\}$ and therefore $I = \{0\}$, because A is semi-prime. Hence $T^2x = 0$ implies $Tx = 0$, which shows that $\text{Ker}(T - \lambda)^2 = \text{Ker}(T - \lambda)$ for all $\lambda \in \mathbb{C}$. This completes the proof. □

Note that all semi-simple algebras are semi-prime and that a Gelfand algebra is semi-prime if and only if it contains no non-zero nilpotent element. Thus we have the following.

COROLLARY 7. *Assume that A is a semi-simple Gelfand algebra. Then every multiplier $T \in M(A)$ has finite ascent. Moreover, for any $A \subseteq \mathbb{C}$,*

$$E_T(A) = \bigcap_{\lambda \in \mathbb{C} \setminus A, n \in \mathbb{N}} (T - \lambda)^n(A).$$

By means of Proposition 6 and Corollary 7, it is not hard to prove the generalization of ([4], Proposition 6.2.3).

COROLLARY 8. *Assume that A is a semi-prime Gelfand algebra. Then every multiplier $T \in M(A)$ has the single-valued extension property. Moreover,*

$$A_T(\{0\}) = \{x \in A : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

In [2], Berberian proved that $\widehat{A} := \{\widehat{a} : a \in A\}$ was a full subalgebra of $C(\Delta(A))$, the Banach algebra of all continuous functions on $\Delta(A)$.

PROPOSITION 9. *Assume that A is a semi-simple Gelfand algebra. If \widehat{A} is a supremum norm closed in $C(\Delta(A))$, then every multiplier $T \in M(A)$ has property (β) .*

PROOF. Let X denote the Banach algebra of all Gelfand continuous bounded complex-valued functions on $\Delta(A)$, endowed with the supremum norm. Then X is regular and semi-simple. Let $R : A \rightarrow \widehat{A}$ be the Gelfand transform, that is, $R(a) = \widehat{a}$ for all $a \in A$. Then R is an

isometric isomorphism, by the open mapping theorem. Hence theorem 3.5 of [12] implies that there is a uniquely determined bounded Gelfand continuous function \widehat{T} on $\Delta(A)$ with the property that $\widehat{(Ta)} = \widehat{T}a$ for all $a \in A$. Define the multiplication operator $S \in L(X)$ by $Sf := \widehat{T}f$ for all $f \in X$. Then by Theorem 6.2.6 [4], S is decomposable and $RT = SR$. It follows that T is similar to the restriction of the decomposable operator S to one of its invariant subspaces, which shows that T has property (β) . This completes the proof. \square

Recall from [9] that the maximal ideal space $\Delta(M(A))$ of the multiplier algebra $M(A)$ can be represented as the disjoint union of $\Delta(A)$ and $H(A)$, where $\Delta(A)$ is canonically embedded in $\Delta(M(A))$ and $H(A)$ denotes the hull of A in $\Delta(M(A))$. When $\Delta(A)$ is regarded as a subset of $\Delta(M(A))$, the hull-kernel topology of $\Delta(A)$ coincides with the relative hull-kernel topology induced by $\Delta(M(A))$ and so the Gelfand topology does. Clearly $\Delta(A)$ is hull-kernel and hence Gelfand open in $\Delta(M(A))$. Moreover, it is easily checked that $\Delta(A)$ is always hull-kernel dense in $\Delta(M(A))$. Note that \widehat{T} is the restriction to $\Delta(A)$ of the Gelfand transform of T on $\Delta(M(A))$.

THEOREM 10. *Assume that A is a semi-simple Gelfand algebra. If $T \in M(A)$ has the property (δ) then \widehat{T} is hull-kernel continuous on $\Delta(A)$.*

PROOF. Assume that \widehat{T} is not hull-kernel continuous on $\Delta(A)$. Then there exists a closed subset F of \mathbb{C} such that $\widehat{T}^{-1}(F)$ is not hull-kernel closed in $\Delta(A)$. Let $\phi \in cl(\widehat{T}^{-1}(F)) \setminus \widehat{T}^{-1}(F)$, where $cl(\widehat{T}^{-1}(F)) = h(k(\widehat{T}^{-1}(F)))$ denotes the hull-kernel closure of $\widehat{T}^{-1}(F)$, and let $\lambda := \widehat{T}(\phi)$. Then $\lambda \notin F$. Since $\{\mathbb{C} \setminus \{\lambda\}, \mathbb{C} \setminus F\}$ is an open covering of \mathbb{C} , every $x \in A$ has a decomposition $x = y + z$ such that $\sigma_T(y) \subseteq \mathbb{C} \setminus \{\lambda\}$ and $\sigma_T(z) \subseteq \mathbb{C} \setminus F$. It follows from $\lambda \in \rho_T(y) \subseteq \rho(T)$ that there exists $u \in A$ such that $y = (T - \lambda)u$, where $\rho_T(y) := \mathbb{C} \setminus \sigma_T(y)$ is the local resolvent of T at the point y . Thus $\phi(y) = \phi(Tu) - \lambda\phi(u) = \widehat{T}(\phi)\phi(u) - \lambda\phi(u) = 0$. Let $\psi \in \widehat{T}^{-1}(F)$. Then $\mu := \widehat{T}(\psi) \in F \subseteq \rho_T(z) \subseteq \rho(T)$, and so there exists $v \in A$ such that $z = (T - \psi)v$, which implies that $\psi(z) = 0$. Thus $z \in k(\widehat{T}^{-1}(F))$ and since $\psi \in h(k(\widehat{T}^{-1}(F)))$, we conclude that $\psi(z) = 0$. It follows that the functional $\psi \in \Delta(A)$ vanishes identically on A , which

is impossible. This contradiction shows that \widehat{T} is hull-kernel continuous on $\Delta(A)$. \square

COROLLARY 11. *Assume that A is a regular, semi-simple Gelfand algebra, and let $T \in M(A)$. Then \widehat{T} is hull-kernel continuous on $\Delta(A)$.*

If $M(A)$ is regular then A is regular ([9], Theorem 1.4.4), but unfortunately the converse is not true in general. A classical counterexample is $A := L_1(G)$, G a nondiscrete locally compact abelian group. Then $M(A) \cong M(G)$, the convolution algebra of all bounded regular complex Borel measures on G .

COROLLARY 12. *Assume that A is a commutative C^* -algebra, and let $T \in M(A)$. Then \widehat{T} is continuous on $\Delta(A)$ with respect to Gelfand topology.*

PROOF. Theorem 4 and Corollary 11. \square

LEMMA 13 ([6]). *Assume that A is a Gelfand algebra, and let $T \in M(A)$ and consider an analytic function $f : U \rightarrow \mathbb{C}$ on an open connected neighborhood U of $\sigma(T)$. If T has property (δ) , then $f(T)$ has property (δ) .*

LEMMA 14. *Assume that A is a Gelfand algebra. If $T \in M(A)$ and $S \in M(A)$ have property (δ) , then $T + S$ and TS have property (δ) .*

PROOF. Assume that T and S have the property (δ) . It is easily checked that $T + S$ has the property (δ) . From Lemma 13, it follows that $f(\lambda, \mu) := \exp(\lambda S + \mu T)$ has the property (δ) for all $\lambda, \mu \in \mathbb{C}$. Since $TS = ST$, we have $ST = \partial_\lambda \partial_\mu f(\lambda, \mu)(0, 0)$. Hence ST has the property (δ) . This completes the proof. \square

THEOREM 15. *Assume that A is a Gelfand algebra. Then*

$$M_D(A) := \{T \in M(A) : T \text{ has the property } (\delta)\}$$

is a closed full commutative subalgebra of $L(A)$.

PROOF. It follows from Lemma 14 that $M_D(A)$ is a closed commutative subalgebra of the Banach algebra $L(A)$. Assume that $T \in M_D(A)$ is bijective. It follows from theorem 1.1.3 of [9] that T^{-1} is a multiplier on A . Since Lemma 13 may be applied to the analytic function f given by $f(\lambda) := \lambda^{-1}$ for all $\lambda \neq 0$, we have T^{-1} has property (δ) . This shows that $M_D(A)$ is a closed full commutative subalgebra of $L(A)$. \square

LEMMA 16. *Let A denote a semi-simple Gelfand algebra and let B denote a regular semi-simple Gelfand algebra. Consider a multiplier $T \in M(A)$ and a multiplication operator $T_a \in L(B)$ for some $a \in B$. Then every linear transformation $\Theta : A \rightarrow B$ with $\Theta T = T_a \Theta$ satisfies*

$$\Theta A_T(F) \subseteq B_{T_a}(F) \quad \text{for all closed } F \subseteq \mathbb{C}.$$

PROOF. It follows from proposition 1.1.2 of [4] that $\sigma(T|_{A_T(F)}) \subseteq F$, since T has the single-valued extension property. Thus $A_T(F) \subseteq E_T(F)$ for all closed $F \subseteq \mathbb{C}$, and so $\Theta A_T(F) \subseteq \Theta E_T(F) = \Theta(T - \lambda)E_T(F) = (T_a - \lambda)\Theta E_T(F)$ for every $\lambda \in \mathbb{C} \setminus F$. This shows that $\Theta A_T(F) \subseteq \Theta E_T(F) \subseteq E_{T_a}(F) = B_{T_a}(F)$. \square

An operator $T \in L(X)$ is said to be a *Riesz operator* if for each $\lambda \in \mathbb{C} \setminus \{0\}$ both the dimension of the kernel $\text{Ker}(T - \lambda I)$ and the codimension of the range $(T - \lambda I)(X)$ in X are finite. A Riesz operator on a Banach space X is a bounded operator whose spectrum has a similar structure to that of a compact operator and its non-zero spectral points are all poles of the resolvent $R(\lambda, T) := (\lambda I - T)^{-1}$. Recall that a complex number $\lambda \in \mathbb{C}$ is a *critical eigenvalue* of the pair (T, S) if λ is an eigenvalue of S and if the codimension of $(T - \lambda I)(X)$ in X is infinite.

THEOREM 17. *Assume that A is a regular semi-simple Gelfand algebra. If $T \in M(A)$ is a Riesz operator and if a is not a divisor of zero in A , then every linear transformation $\Theta : A \rightarrow A$ with $\Theta T = T_a \Theta$ is continuous.*

PROOF. Since a Riesz operators has a countable spectrum, it follows from proposition 2.2 of [4] that T is decomposable. Also, T_a is super decomposable, since A is regular semi-simple. Since 0 is not in the point spectrum of T_a , the pair (T, T_a) has no critical eigenvalue. It remains to

show that T_a has no divisible subspace different from $\{0\}$. If $Z \subseteq A$ is T_a -divisible subspace then

$$Z = \bigcap_{\lambda \in \mathbb{C}} (T_a - \lambda I)Z \subseteq \bigcap_{\lambda \in \mathbb{C}} (T_a - \lambda I)(A) \subseteq \text{rad}(A),$$

where $\text{rad}(A)$ denotes the radical of A . The semi-simplicity of A implies that

$$Z = \bigcap_{\lambda \in \mathbb{C}} (T_a - \lambda I)(A) = \{0\},$$

this implies that T_a has no divisible subspace. It follows from theorem 4.3 of [11] that Θ is continuous, which completes the proof. \square

References

- [1] E. Albrecht and J. Eschmeier, *Analytic functional models and local spectral theory (submitted)*.
- [2] S. K. Berberian, *Lectures in functional analysis and operator theory*, Springer-Verlag, New York (1973).
- [3] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York (1973).
- [4] I. Colojoara and C. Foias, *Theory of generalized spectral operators*, Gordon and Breach, New York (1968).
- [5] J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York (1985).
- [6] I. Erdelyi and R. Lange, *Spectral decompositions on Banach spaces, Lecture notes in mathematics*, Springer-Verlag (1977).
- [7] S. Helgason, *Multipliers of Banach algebras*, Ann. of Math. **64** (1956), 204-254.
- [8] H. Heuser, *Functional analysis*, New York (1982).
- [9] R. Larsen, *An introduction to the theory of multipliers*, Springer-Verlag, New York (1971).
- [10] K. B. Laursen, *Operators with finite ascent*, Pacific J. Math. **152** (1992), 326-336.
- [11] K. B. Laursen and M. M. Neumann, *Decomposable operators and automatic continuity*, J. Operator Theory **15** (1986), 33-51.
- [12] J. K. Wang, *Multipliers of commutative Banach algebras*, Pacific J. Math. **11** (1961), 1131-1149.
- [13] J. G. Wendel, *On isometric isomorphism of group algebras*, Pacific J. Math. **1** (1951), 305-311.

Department of Liberal Arts and Science
 Chodang University
 Muan 534-800, Korea