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A SIX-POINT CHARACTERIZATION OF HILBERT SPACES

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ABSTRACT. A characterization of Hilbert spaces is given in terms of four boundary points and two interior points of the unit sphere.

1. Introduction

The goal of this paper is to present a characterization of Hilbert spaces in terms of four boundary points and two interior points of the unit sphere.

Suppose that $\mathbf{X}$ is a Banach space with norm $\|\cdot\|$. Then we obtain the following characterization of Hilbert spaces:

THEOREM. *A Banach space* $\mathbf{X}$ *is a Hilbert space if and only if*

$$2 \le \frac{|u_2 - x|}{|u_2 - u_1|}\,|u_1 + y| + \frac{|u_1 - x|}{|u_2 - u_1|}\,|u_2 + y| + \frac{|v_2 - x|}{|v_2 - v_1|}\,|v_1 - y| + \frac{|v_1 - x|}{|v_2 - v_1|}\,|v_2 - y|,$$

for any x, y *in* $\mathbf{X}$ *with* $|x| < 1$, $|y| < 1$, *and any* u_1, u_2, v_1, v_2 *in the unit sphere* S_X *of* $\mathbf{X}$ *so that*

$$\{x\} = [u_1, u_2] \cap [v_1, v_2].$$

Here $[u, v]$ *denotes the line segment joining* u *and* v.

2. Lemmas

To prove the theorem, we will use the following geometrical characterization of Hilbert spaces given by Burkholder [2, 3]:

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LEMMA 1. *A Banach space* $\mathbf{X}$ *is a Hilbert space if and only if there is a biconvex function* $\zeta : \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ *such that* $\zeta(0,0) = 1$ *and*

$$\zeta(x,y) \leq |x+y| \quad \textit{if} \quad |x| \vee |y| \geq 1. \tag{1}$$

For a Hilbert space $\mathbf{H}$, Burkholder [3] showed that the function ζ_H given below satisfies all the conditions imposed in Lemma 1:

$$\begin{aligned}\zeta_H(x,y) &= (1 + 2\,(x,y) + |x|^2|y|^2)^{\frac{1}{2}} \quad \text{if} \quad |x| \vee |y| < 1, \\ &= |x+y| \quad \text{if} \quad |x| \vee |y| \geq 1.\end{aligned}$$

Here (x,y) is the real part of the inner product of x and y.

Replacing $\zeta(x,y)$ by the maximum of $\zeta(x,y)$, $\zeta(y,x)$, $\zeta(-x,-y)$, and $\zeta(-y,-x)$, we can assume that the function ζ satisfies the symmetry property:

$$\zeta(x,y) = \zeta(y,x) = \zeta(-x,-y). \tag{2}$$

We also need the following lemmas from the theory of convex bodies. See [1] or [4].

LEMMA 2. *Suppose that* $\mathbf{X}$ *is a two-dimensional real Banach space. Then the norm of* $\mathbf{X}$ *is generated by an inner product if and only if the unit sphere of* $\mathbf{X}$ *is an ellipse.*

LEMMA 3. *If* $\mathbf{C}$ *is a symmetric (about the origin) closed convex curve in the plane, then there exists a unique ellipse of maximal area inscribed in* $\mathbf{C}$. *The maximal inscribed ellipse touches* $\mathbf{C}$ *in at least four points which are symmetric pairwise.*

LEMMA 4. *A Banach space* $\mathbf{X}$ *is a Hilbert space if and only if every two dimensional subspace of* $\mathbf{X}$ *is a Hilbert space.*

3. Proof of Theorem

Suppose that $\mathbf{X}$ is a Hilbert space with norm $|\cdot|$. Take x and y in $\mathbf{X}$ with $|x| < 1$ and $|y| < 1$, and u_1, u_2, v_1, and v_2 in the unit sphere S_X of $\mathbf{X}$ so that

$$\{x\} = [u_1, u_2] \cap [v_1, v_2].$$

By the convexity of $\zeta_X(\cdot, 0)$ and (2), we have

$$1 = \zeta_X(0,0) \le \frac{1}{2}\{\zeta_X(x,0) + \zeta_X(-x,0)\} = \zeta_X(x,0).$$

By the biconvexity of ζ_X and (1), we obtain a string of inequalities:

$$\begin{aligned} \zeta_X(x,0) &\le \frac{1}{2}\{\zeta_X(x,y) + \zeta_X(x,-y)\} \\ &\le \frac{1}{2}\{\alpha\,\zeta_X(u_1,y) + (1-\alpha)\,\zeta_X(u_2,y) + \bar{\alpha}\,\zeta_X(v_1,-y) + (1-\bar{\alpha})\,\zeta_X(v_2,-y)\} \\ &\le \frac{1}{2}\{\alpha\,|u_1+y| + (1-\alpha)\,|u_2+y| + \bar{\alpha}\,|v_1-y| + (1-\bar{\alpha})\,|v_2-y|\}. \end{aligned}$$

Here, we have chosen $\alpha = \dfrac{|u_2 - x|}{|u_2 - u_1|}$ and $\bar{\alpha} = \dfrac{|v_2 - x|}{|v_2 - v_1|}$ so that

$$\alpha\, u_1 + (1-\alpha)\, u_2 = x \quad \text{and} \quad \bar{\alpha}\, v_1 + (1-\bar{\alpha})\, v_2 = x.$$

Replace α by $\dfrac{|u_2 - x|}{|u_2 - u_1|}$ and $\bar{\alpha}$ by $\dfrac{|v_2 - x|}{|v_2 - v_1|}$ to obtain the desired result:

$$2 \le \frac{|u_2 - x|}{|u_2 - u_1|}\,|u_1 + y| + \frac{|u_1 - x|}{|u_2 - u_1|}\,|u_2 + y| + \frac{|v_2 - x|}{|v_2 - v_1|}\,|v_1 - y| + \frac{|v_1 - x|}{|v_2 - v_1|}\,|v_2 - y|.$$

For the converse, suppose that $\mathbf{X}$ is not a Hilbert space. Then we will find x, y in $\mathbf{X}$ with $|x| < 1$, $|y| < 1$ and u_1, u_2, v_1, v_2 in the unit sphere of $\mathbf{X}$ with $\{x\} = [u_1, u_2] \cap [v_1, v_2]$ for which the inequality in Theorem fails.

By Lemma 4, we can assume that the dimension of $\mathbf{X}$ is equal to two. Denote the norm of $\mathbf{X}$ by $|\cdot|$. Let S_X be the unit sphere of $\mathbf{X}$ with respect to $|\cdot|$, that is, $S_X = \{x \in \mathbf{X} : |x| = 1\}$.

By Lemma 3 due to Loewner, there is an ellipse S_0 of maximal area inscribed in S_X with at least four contact points which are symmetric pairwise. Denote by $\|\cdot\|$ the norm induced by S_0 so that $S_0 = \{x \in \mathbf{X} : \|x\| = 1\}$. After some affine transformations, we can assume that S_0 is the unit circle. Let $\pm A$ and $\pm C$ be four distinct contact points with no contact points in the interior of the arc $\overset{\frown}{AC}$.

Let $\theta = \frac{1}{2}\angle AOC$, one half of the angle determined by the line segments $\overline{OA}$ and $\overline{OC}$. Here O denotes the origin of $\mathbf{X}$. We can assume that $0 < 2\theta \leq \pi/2$, $A = (1,0)$, and $C = (\cos 2\theta, \sin 2\theta)$. Observe that the point $(\cos\theta, \sin\theta)$ belongs to S_0, and so it lies inside of S_X. Thus there is a real number $s > 1$ satisfying $|s\,(\cos\theta, \sin\theta)| = 1$.

Let

$$x = \frac{A+C}{2} = \cos\,\theta\,(\cos\,\theta, \sin\,\theta),$$
$$y = y(t) = t\,(\cos\,\theta, \sin\,\theta), \quad t \in (-s, s).$$

Let

$$u_1 = s\,(\cos\,\theta, \sin\,\theta), \qquad u_2 = -s\,(\cos\,\theta, \sin\,\theta),$$

and

$$v_1 = (1,0), \qquad v_2 = (\cos 2\theta, \sin 2\theta).$$

Let

$$\gamma = \frac{|u_2 - x|}{|u_2 - u_1|} = \frac{s + \cos\theta}{2s} \quad \text{and} \quad \overline{\gamma} = \frac{|v_2 - x|}{|v_2 - v_1|} = \frac{1}{2}.$$

Clearly we have $|x| < 1$, $|y| < 1$, and $|u_i| = |v_i| = 1$, for $i = 1, 2$. Since $\gamma\,u_1 + (1-\gamma)\,u_2 = x$ and $\overline{\gamma}\,v_1 + (1-\overline{\gamma})\,v_2 = x$, we also have $\{x\} = [u_1, u_2] \cap [v_1, v_2]$. Let X and Y be simple functions defined on $[0,1)$ by

$$X = u_1\,I_{[0,\frac{\gamma}{2})} + u_2\,I_{[\frac{\gamma}{2},\frac{1}{2})} + v_1\,I_{[\frac{1}{2},\frac{\overline{\gamma}+1}{2})} + v_2\,I_{[\frac{\overline{\gamma}+1}{2},1)},$$
$$Y = Y(t) = y(t)\,I_{[0,\frac{1}{2})} - y(t)\,I_{[\frac{1}{2},1)}.$$

Let f and g be functions defined on an interval $(-s, s)$ by

$$\begin{aligned} f(t) &= 2\,E\,|X + Y(t)| \\ &= 1 + \frac{t}{s^2}\,\cos\,\theta + |(1 - t\,\cos\,\theta, -t\,\sin\,\theta)|\,\overline{\gamma} \\ &\quad + |(\cos\,2\theta - t\,\cos\,\theta, \sin\,2\theta - t\,\sin\,\theta)|(1 - \overline{\gamma}), \\ g(t) &= 1 + \frac{t}{s^2}\,\cos\,\theta + \|(1 - t\,\cos\,\theta, -t\,\sin\,\theta)\|\,\overline{\gamma} \\ &\quad + \|(\cos\,2\theta - t\,\cos\,\theta, \sin\,2\theta - t\,\sin\,\theta)\|\,(1 - \overline{\gamma}). \end{aligned}$$

Then, for $t \in (-s, s)$,

$$f(t) \le g(t) \quad \text{with } f(0) = g(0) = 2,$$
$$g(t) = 1 + \frac{t}{s^2} \cos\theta + (1 - 2t\cos\theta + t^2)^{1/2},$$
$$g'(t) = \frac{\cos\theta}{s^2} + \frac{-\cos\theta + t}{(1 - 2t\cos\theta + t^2)^{1/2}}.$$

In particular,

$$g'(0) = \frac{\cos\theta}{s^2} - \cos\theta < 0 \quad \text{since } s > 1,\ \pi/4 \ge \theta > 0.$$

So there is an $\epsilon > 0$ so that $f(t) \le g(t) < 2$ for $t \in (0, \epsilon)$; therefore

$$2E|X + Y(t)| = \gamma\,|u_1 + y(t)| + (1-\gamma)\,|u_2 + y(t)| + \overline{\gamma}\,|v_1 - y(t)| + (1-\overline{\gamma})\,|v_2 - y(t)| < 2.$$

Replacing γ by $\dfrac{|u_2 - x|}{|u_2 - u_1|}$ and $\overline{\gamma}$ by $\dfrac{|v_2 - x|}{|v_2 - v_1|}$, we obtain

$$2 > \frac{|u_2 - x|}{|u_2 - u_1|}\,|u_1 + y| + \frac{|u_1 - x|}{|u_2 - u_1|}\,|u_2 + y| + \frac{|v_2 - x|}{|v_2 - v_1|}\,|v_1 - y| + \frac{|v_1 - x|}{|v_2 - v_1|}\,|v_2 - y|.$$

This completes the proof of Theorem.

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