Comm. Korean Math. Soc. **12** (1997), No. 4, pp. 895–904

JULIA OPERATORS AND LINEAR SYSTEMS

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ABSTRACT. Let B(z) be a power series with operator coefficients where multiplication by B(z), T, is a contractive and everywhere defined transformation in the square summable power series. Then there is a Julia operator U for T such that

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}),$$

where $\mathcal{D}$ is the state space of a conjugate canonical linear system with transfer function B(z).

1. Linear systems

A vector space $\mathcal{K}$ over the complex numbers with a scalar product $\langle\ ,\ \rangle_{\mathcal{K}}$ is called a Krein space if $\mathcal{K}$ is an orthogonal sum of a Hilbert space $\mathcal{K}_+$ and the anti-space of a Hilbert space $\mathcal{K}_-$. In genenal, such decompositions are not unique. The choice of orthogonal decomposition induces a Hilbert space strong topology on $\mathcal{K}$. The strong topology of this Hilbert space is called the Mackey topology of $\mathcal{K}$. The norm of the Hilbert space depends on the choice of orthogonal decomposition, but two such norms are equivalent.

Let $\mathcal{H}$ and $\mathcal{C}$ be Krein spaces. A continuous linear transformation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \oplus \mathcal{C} \longrightarrow \mathcal{H} \oplus \mathcal{C}$$

is called a linear system. The underlying Krein space $\mathcal{H}$ is called the state space and the auxiliary Krein space $\mathcal{C}$ is called the coefficient space

Received February 17, 1997. Revised August 30, 1997.
1991 Mathematics Subject Classification: 47B37.
Key words and phrases: Julia operator, conjugate isometry linear system.
This paper was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation.

or the external space. The transformation A is called the main transformation. The transformation B is called the input transformation. The transformation C is called the output transformation. The operator is called the external operator.

A linear system is said to be contractive if the matrix is contractive, unitary if the matrix is unitary, and conjugate isometric if the matrix has an isometric adjoint. The transfer function $W(z)$ of the linear system is defined by

$$W(z) = D + zC(I - zA)^{-1}B.$$

A linear system is said to be observable if there is no nonzero element f of the state space such that $CA^n f = 0$ for every nonnegative integer n. An observable linear system is said to be in a canonical form if the elements of the state space are power series with vector coefficients in such a way that the identity $a_n = CA^n f$ holds whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If an observable linear system is in a canonical form, then the elements of the state space are power series which converge in some neighborhood of the origin. For this linear system the main transformation $A(f(z)) = [f(z) - f(0)]/z$, $B(c) = [W(z) - W(0)]c/z$, $C(f(z)) = f(0)$, and $D(c) = W(0)c$, where $W(z)$ is the transfer function of the linear system.

The theory of canonical linear systems which are conjugate isometric is a generalization of the theory of square summable power series with vector coefficients. Assume that the coefficient space $\mathcal{C}$ is a Krein space. Write $\mathcal{C}$ as the orthogonal sum of a Hilbert space $\mathcal{C}_+$ and the anti-space $\mathcal{C}_-$ of a Hilbert space. Let J be the operator which is the identity on $\mathcal{C}_+$ and which is minus the identity on $\mathcal{C}_-$. If b is any vector b^- denotes the linear functional on vectors defined by the scalar product $b^- a = < a, b >_{\mathcal{C}}$. Let

$$\mathcal{C}(z) = \{f : f(z) = \sum_{n=0}^{\infty} a_n z^n, a_n \in \mathcal{C}, \sum_{n=0}^{\infty} a_n^- J a_n < \infty\}.$$

The condition does not depend on the choice of decompositions of $\mathcal{C}$. The space $\mathcal{C}(z)$ is considered as a Krein space with the unique scalar product such that

$$\langle f(z), f(z) \rangle_{\mathcal{C}(z)} = \sum_{n=0}^{\infty} a_n^- a_n.$$

The identity for difference-quotients

$$\langle [f(z) - f(0)]/z, [g(z) - g(0)]/z \rangle_{\mathcal{C}(z)} = \langle f(z), g(z) \rangle_{\mathcal{C}(z)} - \langle f(0), g(0) \rangle_{\mathcal{C}}$$

holds for all $f(z)$ and $g(z)$ in $\mathcal{C}(z)$. These properties imply that the space $\mathcal{C}(z)$ is the state space of a canonical linear system which is conjugate isometric and has transfer function identically zero.

2. Complementary theory and Julia operators

The construction of linear systems in Krein spaces makes use of a Krein space generalization of de Branges' complementary theory [2, 3].

If a Krein space $\mathcal{P}$ is contained continuously and contractively in a Krein space $\mathcal{H}$, then a unique Krein space $\mathcal{Q}$ exists, which is contained continuously and contractively in $\mathcal{H}$, such that the inequality

$$\langle c, c \rangle_{\mathcal{H}} \ \leq \ \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some such decomposition for which equality holds. The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. A unique minimal decomposition is obtained when equality holds. If

$$\langle c, c \rangle_{\mathcal{H}} \ = \ \langle a, a \rangle_{\mathcal{P}} + \langle b, b \rangle_{\mathcal{Q}}$$

where $c = a + b$, then a is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and b is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. If $\mathcal{P}$ is contained continuously and isometrically in $\mathcal{H}$, then $\mathcal{Q}$ is the orthogonal complement of $\mathcal{P}$ in $\mathcal{H}$. Complementation theory can be used to give new proofs of theorems of Dritschel [8] and of Dritschel and Rovnyak [7] which generalize the commutant lifting theorem to Krein spaces [5]. There is a close relation between a Julia operator and a linear system.

Let $\mathbf{B}(\mathcal{H}, \mathcal{K})$ be the set of continuous linear transformation of a Krein space $\mathcal{H}$ into a Krein space $\mathcal{K}$ and $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$. An operator $\tilde{D}$ from a Krein space $\tilde{\mathcal{D}}$ to a Krein space $\mathcal{H}$ is called a defect operator for T if $\tilde{D}$ has zero kernel and $1 - T^*T = \tilde{D}\tilde{D}^*$.

By a Julia operator for T we mean any unitary operator of the form

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}),$$

where $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are Krein spaces and $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ and $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ are operators with zero kernels. In this case $\tilde{D}$ is a defect oerator for T, D is a defect operator for T^* and U^* is a Julia operator for T^*.

The existence of a Julia operator is based on the factorization of a self-adjoint transformation.

THEOREM 2.1. ([7], Theorem 1.2.2) *Let $\mathcal{H}$ be a Krein space, and let $H \in \mathbf{B}(\mathcal{H})$ be a self-adjoint operator. Then there is a Krein space $\mathcal{A}$ and operator $A \in \mathbf{B}(\mathcal{A}, \mathcal{H})$ with zero kernel such that $H = AA^*$.*

The existence of a defect operator for any given operator T implies the existence of a Julia operator for any given operator T.

THEOREM 2.2. ([7], Theorem 1.2.4) *Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces $T \in \mathbf{B}(\mathcal{H}, \mathcal{K})$. If $\tilde{D} \in \mathbf{B}(\tilde{\mathcal{D}}, \mathcal{H})$ is a defect operator for T, there exists a Julia operator of the form*

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{H} \oplus \mathcal{D}, \mathcal{K} \oplus \tilde{\mathcal{D}}).$$

The charactorization of the range of a contractive operator is given by Dritschel.

THEOREM 2.3. ([8], Theorem 11) *Let T be a contractive transformation on a Krein space $\mathcal{H}$ to a Krein space $\mathcal{K}$, and let $D \in \mathbf{B}(\mathcal{D}, \mathcal{K})$ be a defect operator for T^*. Then an element g of $\mathcal{K}$ belongs to the range of T if and only if*

$$\sup_{u \in \mathcal{D}} [< g + Du, g + Du >_{\mathcal{K}} - < u, u >_{\mathcal{D}}] < \infty.$$

Let $B(z)$ be a power series with operator coefficients such that multiplication by $B(z)$, T is one-to-one in $\mathcal{C}(z)$. Let $\mathcal{M}(B)$ be the range of T in the scalar product which makes T an isometry of $\mathcal{C}(z)$ onto $\mathcal{M}(B)$.

Then $M(B)$ is contained continuously in $\mathcal{C}(z)$ and the adjoint of the inclusion of $\mathcal{M}(B)$ in $\mathcal{C}(z)$ coincides with TT^*. Let $\mathcal{H}(B)$ be any Krein space which is contained continuously in $\mathcal{C}(z)$ such that the adjoint of inclusion of $\mathcal{H}(B)$ in $\mathcal{C}(z)$ coincides with $1-TT^*$. Define the overlapping space $\mathcal{L}$ of $\mathcal{H}(B)$ with respect to $\mathcal{M}(B)$ is the set of $f(z)$ in $\mathcal{C}(z)$ with $B(z)f(z)$ in $\mathcal{H}(B)$. Consider $\mathcal{L}$ as a Krein space with the scalar product such that the identity

$$\langle f(z), g(z)\rangle_{\mathcal{L}} = \langle B(z)f(z), B(z)g(z)\rangle_{\mathcal{M}(B)} + \langle f(z), g(z)\rangle_{\mathcal{H}(B)}$$

holds for every $f(z)$ and $g(z)$ in $\mathcal{L}$. Dritschel [8] shows that there is a Julia operator for T which is given by

$$U = \begin{pmatrix} T & D \\ \tilde{D}^* & L \end{pmatrix} \in \mathbf{B}(\mathcal{C}(z) \oplus \mathcal{D}, \mathcal{C}(z) \oplus \tilde{\mathcal{D}}),$$

where $\mathcal{D} = \mathcal{H}(B)$ is a Krein space and D is the inclusion operator, $\tilde{\mathcal{D}} = \mathcal{L}$ is a Krein space and $\tilde{D}$ is the inclusion operator, and $L \in \mathbf{B}(\mathcal{D}, \tilde{\mathcal{D}})$ is the operator such that $L^*f = -Tf$ for $f \in \tilde{\mathcal{D}}$.

The complementation theory can be generalized for any self-adjoint operator.

THEOREM 2.4. *Assume that $B(z)$ is a power series with operator coefficients such that multiplication by $B(z)$ in $\mathcal{C}(z)$ is one-to-one. Then a Krein space $\mathcal{D}$ exists such that $\mathcal{D}$ is contained continuously in $\mathcal{C}(z)$ and there is a partial isometry from $\mathcal{D} \times \mathcal{C}(z)$ to $\mathcal{C}(z)$ which takes $(f(z), g(z))$ into $f(z) + B(z)g(z)$.*

PROOF. Let T be multiplication by $B(z)$ in $\mathcal{C}(z)$ and let U a Julia operator for T as above. We have to show that the identity

$$\begin{aligned} &< f(z) + B(z)g(z), f(z) + B(z)g(z) >_{\mathcal{C}(z)} \\ &=< f(z), f(z) >_{\mathcal{D}} + < g(z), g(z) >_{\mathcal{C}(z)} \end{aligned}$$

holds for any $f(z)$ in $\mathcal{D}$ and $g(z)$ in $\mathcal{C}(z)$ if, and only if, the identity

$$< f(z), B(z)h(z) >_{\mathcal{D}} = < g(z), h(z) >_{\mathcal{C}(z)}$$

holds for every $B(z)h(z)$ in $\mathcal{D}$.

Since U is unitary,

$$T^*T + \tilde{D}\tilde{D}^* = 1, \quad TT^* + DD^* = 1,$$
$$D^*T + L^*\tilde{D}^* = 0, \quad T\tilde{D} + DL^* = 0,$$
$$D^*D + L^*L = 1, \quad \tilde{D}^*\tilde{D} + LL^* = 1.$$

Let $f(z)$ be in $\mathcal{D}$ and $g(z)$ in $\mathcal{C}(z)$ such that the identity

$$< f(z), B(z)h(z) >_{\mathcal{D}} = < g(z), h(z) >_{\mathcal{C}(z)}$$

holds for every $B(z)h(z)$ in $\mathcal{D}$. The identity

$$\begin{aligned} < f(z), f(z) >_{\mathcal{D}} &= < L^*Lf(z), f(z) >_{\mathcal{D}} + < D^*Df(z), f(z) >_{\mathcal{D}} \\ &= - < TLf(z), f(z) >_{\mathcal{D}} + < Df(z), Df(z) >_{\mathcal{C}(z)} \\ &= - < Lf(z), g(z) >_{\mathcal{C}(z)} + < f(z), f(z) >_{\mathcal{C}(z)} \end{aligned}$$

is satisfied. Since the identity

$$\begin{aligned} < Lf(z), g(z) >_{\mathcal{C}(z)} &= < Lf(z), \tilde{D}\tilde{D}^*g(z) >_{\tilde{\mathcal{D}}} \\ &= < f(z), L^*\tilde{D}\tilde{D}^*g(z) >_{\mathcal{D}} \\ &= - < g(z), \tilde{D}\tilde{D}^*g(z) >_{\mathcal{C}(z)} \\ &= - < g(z), g(z) >_{\mathcal{C}(z)} + < Tf(z), Tf(z) >_{\mathcal{C}(z)} \end{aligned}$$

holds, the identity

$$\begin{aligned} &< f(z) + B(z)g(z), f(z) + B(z)g(z) >_{\mathcal{C}(z)} \\ &= < f(z), f(z) >_{\mathcal{D}} + < g(z), g(z) >_{\mathcal{C}(z)} \end{aligned}$$

is satisfied. This completes the proof of the theorem. □

3. Existence of linear systems

Let $\mathcal{H}$ be the state space of a canonical linear system which is conjugate isometric with transfer function $B(z)$. The augumented space $\mathcal{H}'$ is the set of power series $f(z)$ with vector coefficients such that

$[f(z) - f(0)]/z$ belongs to $\mathcal{H}$. The space $\mathcal{H}'$ becomes a Krein space when considered with the Cartesian scalar porduct Krein space $\mathcal{H} \times \mathcal{C}$. This is the unique scalar product for which the identity for the difference-quotients

$$< f(z), f(z) >_{\mathcal{H}'} = < [f(z) - f(0)]/z, [f(z) - f(0)]/z >_{\mathcal{H}} + f(0)^{-} f(0)$$

holds for every element $f(z)$ in $\mathcal{H}'$. The space $\mathcal{H}'$ is the state space of a canonical linear system which is conjugate isometric with transfer function $zB(z)$. In this notation the matrix of the canonical linear system with the state space $\mathcal{H}$ and transfer function $B(z)$ is isometric to the transformation of $\mathcal{H}'$ into itself which takes $f(z)$ into $[f(z) - f(0)]/z + B(z)f(0)$. An equivalent condition is that a patially isometric transformation of the Cartisian product Krein space $\mathcal{H} \times \mathcal{C}$ onto $\mathcal{H}'$ is defined by taking a pair $(f(z), c)$ into $f(z) + B(z)c$. Explicitly this means that $\mathcal{H}$ is contained continuously in $\mathcal{H}'$ and that multiplication by $B(z)$ is a continuous transformation of $\mathcal{C}$ into $\mathcal{H}'$.

Let S be multiplication by $B'(z) = zB(z)$ in $\mathcal{C}(z)$. If T is one-to-one, then S is also one-to-one. Hence there is a Julia operator for S which is given by

$$U_1 = \begin{pmatrix} S & D_1 \\ \tilde{D}_1^* & L_1 \end{pmatrix} \in \mathbf{B}(\mathcal{C}(z) \oplus \mathcal{D}_1, \mathcal{C}(z) \oplus \tilde{\mathcal{D}}_1),$$

where $\mathcal{D}_1 = \mathcal{H}(B')$ is a Krein space and D_1 is the inclusion operator, $\tilde{\mathcal{D}}_1 = \tilde{\mathcal{L}}_1$ is a Krein space and $\tilde{D}_1$ is the inclusion operator, and $L_1 \in \mathbf{B}(\mathcal{D}_1, \tilde{\mathcal{D}}_1)$ is the operator such that $L_1^* f = -Sf$ for $f \in \tilde{\mathcal{D}}_1$. Since $1 - T^*T = 1 - S^*S$, the adjoint of the inclusion of $\tilde{\mathcal{D}}$ in $\mathcal{C}(z)$ coincides with the adjoint of the inclusion of $\tilde{\mathcal{D}}_1$ in $\mathcal{C}(z)$.

Assume T is contractive. Then the space $\tilde{\mathcal{D}}$ is a Hilbert space. Theorem 2 implies $\tilde{\mathcal{D}}$ is isometrically equal to $\tilde{\mathcal{D}}_1$. Using the Theorem 5 we can easily show that $[f(z) - f(0)]/z$ is in $\mathcal{D}$ for any f in $\mathcal{D}$ and that $[f(z) - f(0)]/z$ is in $\tilde{\mathcal{D}}$ for any $f(z) \in \tilde{\mathcal{D}}$.

The characterization of $\mathcal{D}_1$ can be made.

THEOREM 3.1. *Assume multiplicaion by $B(z)$ in $\mathcal{C}(z)$, T, is one-to-one and contractive. Then $\mathcal{D}$ is contained in $\mathcal{D}_1$, $[f(z) - f(0)]/z$ is in $\mathcal{D}$ for any $f(z)$ in $\mathcal{D}_1$ and the identity*

$$< f(z), f(z) >_{\mathcal{D}_1} = < [f(z) - f(0)]/z, [f(z) - f(0)]/z >_{\mathcal{D}} + f(0)^{-} f(0)$$

holds for every $f(z)$ in $\mathcal{D}_1$.

PROOF. Let Z be the transformation in $\mathcal{C}(z)$ which takes f(z) into zf(z). Then the adjoint of Z in $\mathcal{C}(z)$, Z^*, takes $f(z)$ into $[f(z)-f(0)]/z$. Let $f(z)$ be in $\mathcal{D}$. Then

$$f(z) = D_1D_1^*Z^*f(z) + ZTT^*Z^*f(z) = D_1D_1^*f(z) + L_1^*Z^*Lf(z) \in \mathcal{D}_1$$

since $Z^*(\tilde{\mathcal{D}}) \subset \tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}$ is isometrically equal to $\tilde{\mathcal{D}}_1$. Let $f(z)$ be in $\mathcal{D}_1$

$$\begin{aligned} Z^*f(z) &= Z^*D_1f(z) = DD^*Z^*f(z) + TT^*Z^*D_1f(z) \\ &= DD^*Z^*D_1f(z) - T\tilde{D}_1L_1f(z) \end{aligned}$$

Since $T\tilde{D}_1(\tilde{\mathcal{D}}_1) \subset \mathcal{D}$, $Z^*f(z)$ is in $\mathcal{D}$. The identity

$$\begin{aligned} &< Z^*f(z), Z^*f(z) >_{\mathcal{D}} \\ &=< DD^*Z^*D_1f(z), Z^*f(z) >_{\mathcal{D}} - < T\tilde{D}_1L_1f(z), Z^*f(z) >_{\mathcal{D}} \\ &=< Z^*f(z), Z^*f(z) >_{\mathcal{C}(z)} - < T\tilde{D}_1L_1f(z), Z^*f(z) >_{\mathcal{D}} \\ &=< D_1^*D_1f(z), f(z) >_{\mathcal{D}_1} - f(0)^- f(0) - < T\tilde{D}_1L_1f(z), Z^*f(z) >_{\mathcal{D}} \end{aligned}$$

is satisfied. Since the identity

$$\begin{aligned} &< T\tilde{D}_1L_1f(z), Z^*f(z) >_{\mathcal{D}} \\ &=< T\tilde{D}_1L_1f(z), DD^*Z^*D_1f(z) >_{\mathcal{D}} - < T\tilde{D}_1L_1f(z), T\tilde{D}_1L_1f(z)) >_{\mathcal{D}} \\ &=< T\tilde{D}_1L_1f(z), Z^*D_1f(z) >_{\mathcal{C}(z)} - < \tilde{D}_1L_1f(z), \tilde{D}_1L_1f(z)) >_{\tilde{\mathcal{D}}} \\ &\quad + < \tilde{D}_1L_1f(z), \tilde{D}_1L_1f(z) >_{\mathcal{C}(z)} \\ &=< TL_1f(z), Z^*f(z) >_{\mathcal{C}(z)} - < L^*L_1f(z), f(z) >_{\tilde{D}} \\ &\quad + < L_1f(z), L_1f(z) >_{\mathcal{C}(z)} \\ &= - < L^*L_1f(z), f(z) >_{\tilde{D}} \end{aligned}$$

hold, we have

$$< Z^*f(z), Z^*f(z) >_{\mathcal{D}} =< f(z), f(z) >_{\mathcal{D}_1} - < f(0), f(0) >_{\mathcal{C}} .$$

This completes the proof of the theorem. □

A construction of a conjugate isometric canonical linear system with transfer function $B(z)$ can be made.

THEOREM 3.2. *There is a partial isometry from $\mathcal{D}\times\mathcal{C}$ onto $\mathcal{D}_1$ which takes $(f(z),c)$ into $f(z)+B(z)c$.*

PROOF. Let A be the transformation on $\mathcal{D}$ which takes $f(z)$ into $[f(z)-f(0)]/z$. Since the identity

$$ADD^*h(z) = DD^*Z^*h(z)+[B(z)-B(0)]g(0)/z, \quad \text{where } T^*h(z)=g(z)$$

holds for every $h(z)$ in $\mathcal{C}(z)$, the identity

$$\begin{aligned}
&< [f(z)-f(0)]/z, h(z) >_{\mathcal{C}(z)}\\
&=< A^*f(z), DD^*h(z) >_{\mathcal{D}}\\
&=< f(z), DD^*Z^*h(z) >_{\mathcal{D}} - < f(z), [B(z)-B(0)]g(0)/z >_{\mathcal{D}}\\
&=< f(z), Z^*h(z) >_{\mathcal{C}(z)} -g(\bar{0}) < f(z), [B(z)-B(0)]/z >_{\mathcal{D}}\\
&=< zf(z)-B(z)\tilde{f}(0), h(z) >_{\mathcal{C}(z)}
\end{aligned}$$

holds for every $h(z)$ in $\mathcal{C}(z)$ and $f(z)$ in $\mathcal{D}$ where $\tilde{f}(0) =< f(z),[B(z)-B(0)]/z >_{\mathcal{D}}$. This implies that the adjoint of A takes $f(z)$ into $zf(z)-B(z)\tilde{f}(0)$.

Let $u(z)$ be in $\mathcal{D}$ and $v(z)$ in $\mathcal{D}_1$. The identity

$$\begin{aligned}
< u(z),v(z) >_{\mathcal{D}_1} &=< Au(z), [v(z)-v(0)]/z >_{\mathcal{D}} + < u(0), v(0) >_{\mathcal{C}}\\
&=< u(z), A([v(z)-v(0)]/z) >_{\mathcal{D}} + < Du(z), v(0) >_{\mathcal{C}(z)}\\
&=< u(z), A[v(z)-v(0)]/z + D^*v(0) >_{\mathcal{D}}
\end{aligned}$$

implies that the adjoint of inclusion of $\mathcal{D}$ in $\mathcal{D}_1$ takes $f(z)$ into

$$A^*([f(z)-f(0)]/z) + D^*f(0) = f(z) - B(z)[\tilde{Z^*}f(0) + B(0)^-f(0)]$$

where $\tilde{Z^*}f(0) =< [f(z)-f(0)]/z, [B(z)-B(0)]/z >_{\mathcal{D}}$.

Let P be the transformation from $\mathcal{D}\times\mathcal{C}$ onto $\mathcal{D}_1$ which takes $(f(z),c)$ into $f(z)+B(z)c$. Let $(f(z),c)$ be in the orthogonal complement of kernel of P. Since $f(z)$ and $B(z)c$ are in $\mathcal{D}_1$, we can write $f(z)=f(z)-B(z)a+B(z)a$ and $B(z)c=B(z)c-B(z)b+B(z)b$ where $a=\tilde{f}(0)+B(0)^-f(0)$

and $b = \tilde{B}(0) + B(0)^{-}B(0)c$. The adjoint of inclusion of $\mathcal{D}$ in $\mathcal{D}_1$ takes $f(z)$ into $f(z) - B(z)a$ and $B(z)c$ into $B(z)c - B(z)b$. Then the identity

$$\begin{aligned} < f(z), f(z) >_{\mathcal{D}} &=< f(z) - B(z)a, f(z) >_{\mathcal{D}} + < B(z)a, f(z) >_{\mathcal{D}} \\ &=< f(z), f(z) >_{\mathcal{D}_1} + < a, c >_{\mathcal{C}} \\ &=< f(z), f(z) >_{\mathcal{D}_1} + < f(z), B(z)c >_{\mathcal{D}_1} \end{aligned}$$

holds. It implies that the identity

$$\begin{aligned} &< f(z) + B(z)c, f(z) + B(z)c >_{\mathcal{D}_1} \\ &=< f(z), f(z) >_{\mathcal{D}_1} + < B(z)c, f(z) >_{\mathcal{D}_1} \\ &\quad + < f(z), B(z)c >_{\mathcal{D}_1} + < B(z)c, B(z)c >_{\mathcal{D}_1} \\ &=< f(z), f(z) >_{\mathcal{D}} + < B(z)[c-b], f(z) >_{\mathcal{D}} \\ &\quad + < [B(z) - B(0)]c/z, [B(z) - B(0)]c/z >_{\mathcal{D}} + < B(0)c, B(0)c >_{\mathcal{C}} \\ &=< f(z), f(z) >_{\mathcal{D}} + c^{-}(c-b) + < c, \tilde{B}(0)c >_{\mathcal{C}} + < c, B(0)^{-}B(0)c >_{\mathcal{C}} \\ &=< f(z), f(z) >_{\mathcal{D}} + c^{-}c \end{aligned}$$

is satisfied. This completes the proof of the theorem. □

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