

ELLIPTIC SYSTEMS INVOLVING COMPETING INTERACTIONS WITH NONLINEAR DIFFUSIONS II

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ABSTRACT. In this paper, we give sufficient conditions of certain elliptic systems involving competing interactions with nonlinear diffusion rates. The existence of positive solution depends on the sign of the first eigenvalue of operators of Schrödinger type. More precisely, if the sign of such operators are either both positive or both negative, then system has a positive solution. The main tool employed is the fixed point index of compact operator on positive cones.

1. Introduction and Existence Theorem

In this paper, we will investigate the existence of positive solutions to the following elliptic systems representing competing interaction:

$$(1.1) \quad \begin{cases} -\varphi(u, v)\Delta u = uf(u, v) \\ -\psi(u, v)\Delta v = vg(u, v) \\ (u, v) = (0, 0) \end{cases} \quad \begin{array}{l} \text{in } \Omega \\ \text{on } \partial\Omega \end{array}$$

where Ω is a bounded region in \mathbf{R}^n with a smooth boundary and φ, ψ are strictly positive nondecreasing functions. Also u, v represent the densities of certain two species which compete each other. Several results have been obtained for the system (1.1) under Dirichlet or Neumann boundary conditions where the diffusion terms are positive constants, not nonlinear functions. See [4], [8], [9], [10], [11].

It was shown in [2] that the existence of positive solution of the system (1.1) depends on the sign of the first eigenvalue of operator of Schrödinger

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type. i.e., the positive solutions exist if the sign of the first eigenvalues of those operators both are positive.

In this paper, we will show that the positive solutions exists even if the first eigenvalue of the above operators are both negative. The main tool employed is the theorem concerning the fixed point index of compact operator on positive cones.

For the system (1.1) with competing interactions, we expose the following assumptions:

(H1) $f, g \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$f_u(u, v) < 0, \quad f_v(u, v) < 0 \quad \text{for } u, v > 0$$

$$g_u(u, v) < 0, \quad g_v(u, v) < 0 \quad \text{for } u, v > 0$$

$$f(0, 0) > 0, \quad g(0, 0) > 0$$

(H2) There exist positive constants C_1, C_2 such that

$$f(u, 0) < 0 \quad \text{for } u > C_1$$

$$g(0, v) < 0 \quad \text{for } v > C_2$$

(H3) $f(\cdot, v), g(u, \cdot)$ are Lipschits continuous for fixed $u, v \in \mathbb{R}^+$ and concave down where $f(\cdot, v) < 0, g(u, \cdot) < 0$, respectively.

(H4) φ, ψ are strictly positive C^1 -function in u, v , respectively, and nondecreasing, concave down in $u, v \in \mathbb{R}^+$.

Throughout this paper, $\lambda_1(A)$ denote the first eigenvalue of operator A on Ω with homogeneous Dirichlet boundary conditions.

The following lemma appears in [1].

LEMMA 1.1. *Assume that φ is strictly positive, nondecreasing and concave down, and h is monotone nondecreasing C^1 -function with $f(x, 0) > 0$. If $\lambda_1(\varphi(x, 0)\Delta + f(x, 0)) > 0$, then the equation*

$$\begin{cases} -\varphi(x, u)\Delta u = uf(x, u) \\ u = 0 \end{cases} \quad \text{on } \partial\Omega$$

has a unique positive solution in $C^2(\bar{\Omega})$.

By the above lemma, if $\lambda_1(\varphi(0, 0)\Delta + f(0, 0)) > 0$ in addition to (H1)-(H4), then there is a semi-trivial solution $(u_0, 0)$ to (1.1) where u_0 is the positive solution to

$$\begin{cases} \varphi(u)\Delta u + uf(u) = 0 \\ u = 0 \end{cases} \quad \text{on } \partial\Omega$$

Similarly, if $\lambda_1(\psi(0, 0)\Delta + g(0, 0)) > 0$, then there is a semi-trivial solution $(0, v_0)$ to (1.1) where v_0 is the positive solution to

$$\begin{cases} \psi(v)\Delta v + vg(v) = 0 \\ v = 0 \end{cases} \quad \text{on } \partial\Omega$$

In section 3, we show that these solutions $(u_0, 0)$ and $(0, v_0)$ are used to give the sufficient conditions for the existence of positive solutions to the system (1.1).

Now we state the existence theorem of our system (1.1).

THEOREM 1.2. *Suppose that the assumptions (H1)-(H4) hold. Assume that $\lambda_1(\varphi(0, 0)\Delta + f(0, 0)) > 0$ and $\lambda_1(\psi(0, 0)\Delta + g(0, 0)) > 0$.*

(i) *If (u, v) is a strictly positive solution to (1.1), then*

$$0 < u(x) < u_0(x) < C_1, \quad 0 < v(x) < v_0(x) < C_2$$

(ii) *If the first eigenvalues of the operator $\varphi(0, v_0)\Delta + f(0, v_0)I$ and $\psi(u_0, 0)\Delta + g(u_0, 0)I$ are both negative or both positive, then the system (1.1) has a positive solution (u, v) .*

2. Preparations

We state some known lemmas and theorems which will serve as the basic tools in this paper.

Let E be a real Banach space and $W \subset E$ a closed convex set. W is called a *wedge* if $\alpha W \subset W$ for all $\alpha \geq 0$. A wedge is said to be a *cone* if $W \cap (-W) = \{0\}$. For $y \in W$, define

$$W_y = \{x \in E \mid y + \gamma x \in W \text{ for some } \gamma > 0\}$$

$$S_y = \{x \in \overline{W}_y \mid -x \in \overline{W}_y\}.$$

Then \overline{W}_y is a wedge containing W , y , $-y$, while S_y is a closed subspace of E containing y . Let T be a compact linear operator on E which satisfies $T(\overline{W}_y) \subset \overline{W}_y$. We say that T has a *property α* on \overline{W}_y if there is a $t \in (0, 1)$ and a $w \in \overline{W}_y \setminus S_y$ such that $w - tTw \in S_y$.

Let $A : W \rightarrow W$ is a compact operator with fixed point $y \in W$ and A is Fréchet differentiable at y . Let $L = A'(y)$ be the Fréchet derivative of A at y . Then L maps \overline{W}_y into itself.

For an open subset $U \subset W$, define $\text{index}(A, U, W) = \deg_W(I - A, U, 0)$. To have \deg_W well defined we require that W be a retract of E . By a result

of Dugundji, every closed convex subset of real Banach space E is a retract of E . Since W is a wedge in E , W is a retract of E . We also have that S_y is a retract of E . Hence the above index is well defined. If y is an isolated fixed point of A , then the fixed point index of A at y in W is defined by $\text{index}_W(A, y) = \text{index}(A, y, W) = \text{index}(A, U(y), W)$, where $U(y)$ is a small open neighbourhood of y in W . We have the following proposition:

PROPOSITION 2.1. Assume that $I - L$ is invertible on E .

(i) If L has property α on \overline{W}_y , then $\text{index}_W(A, y) = 0$.

(ii) If L does not have property α on \overline{W}_y , then $\text{index}_W(A, y) = (-1)^\sigma$, where σ is the sum of multiplicities of all the eigenvalues of L which are greater than 1.

PROOF. See [5], [9] for the details. \square

Next we state the extended maximum principle. Consider operator $Au := a(x)\Delta u + b(x)u$, $u = 0$ on $\partial\Omega$.

PROPOSITION 2.2. Let $a, b \in L^\infty(\Omega)$. If $\lambda_1(a(x)\Delta + b(x)I) < 0$ holds and $u(x)$ is any nonconstant function satisfying

$$\begin{cases} a(x)\Delta u + b(x)u \geq 0 \\ u = 0 \end{cases} \quad \text{on } \partial\Omega,$$

then $u(x) < 0$ in Ω .

PROOF. One can modify the proof of Lemma 2.2 in [10]. \square

Suppose that φ is strictly positive, nondecreasing and concave down, and f is monotone nonincreasing C^1 -function with $f(0) > 0$. Let u_0 be a unique positive solution to the equation

$$\begin{cases} -\varphi(x, u)\Delta u = uf(x, u) \\ u = 0 \end{cases} \quad \text{on } \partial\Omega.$$

We shall linearize the above equation at $u = u_0 > 0$. Define the solution operator : $S \in C(\bar{\Omega})$ by $S(u) = \bar{u}$, where \bar{u} is the unique solution of

$$\begin{cases} -\varphi(x, \bar{u})\Delta \bar{u} + M\bar{u} = uf(x, u) + Mu \\ \bar{u} = 0 \end{cases} \quad \text{on } \partial\Omega.$$

where $M > 0$ is sufficiently large. Note that $S(u_0) = u_0$. Also we define the operator S_L of linearization by $S_L(w) = v$, where v is the unique

solution of

$$\begin{cases} -\varphi(x, u_0)\Delta v + Mv = wf(x, u_0) + wu_0f_u(x, u_0) + Mw \\ v = 0 \end{cases} \quad \text{on } \partial\Omega.$$

Now we have the following lemma.

LEMMA 2.3. *S is Fréchet differentiable at $u = u_0 \in C(\bar{\Omega})$ and $S'(u_0) = S_L$.*

PROOF. We need to show that

$$\|S(u_0 + w) - S(w) - S_L(w)\| = o(\|w\|)$$

where the norm is taken in $C(\Omega)$. Replace w by u for convenience. Let $\|u\|_\infty$ be small. Denote $\bar{u} = S(u_0 + u)$, and $v = S_L(u)$. Then we have

$$\begin{aligned} -\Delta\bar{u} + M\bar{u}/\varphi(x, \bar{u}) &= (u_0 + u)f(x, u_0 + u)/\varphi(x, \bar{u}) + M(u_0 + u)/\varphi(x, \bar{u}) \\ -\Delta u_0 + Mu_0/\varphi(x, u_0) &= u_0f(x, u_0)/\varphi(x, u_0) + Mu_0/\varphi(x, u_0) \\ -\Delta v + Mv/\varphi(x, u_0) &= uf(x, u_0)/\varphi(x, u_0) + Mu/\varphi(x, u_0) \\ &\quad + uu_0f_u(x, u_0)/\varphi(x, u_0). \end{aligned}$$

From the above three equations

$$\begin{cases} -\varphi(x, u_0)\Delta(\bar{u} - u_0 - v) + M(\bar{u} - u_0 - v) = \varphi(x, u_0)[A - B] \\ (\bar{u} - u_0 - v) = 0 \end{cases} \quad \text{on } \partial\Omega$$

where

$$\begin{aligned} A &= u[f(x, u_0 + u)/\varphi(x, \bar{u}) - f(x, u_0)/\varphi(x, u_0)] + u_0[f(x, u_0 + u)/\varphi(x, \bar{u}) \\ &\quad - f(x, u_0)/\varphi(x, u_0) - uf_u(x, u_0)/\varphi(x, u_0)] + Mu[1/\varphi(x, \bar{u}) - 1/\varphi(x, u_0)] \\ &\quad + Mu_0[1/\varphi(x, \bar{u}) - 1/\varphi(x, u_0)] \\ B &= M\bar{u}[1/\varphi(x, \bar{u}) - 1/\varphi(x, u_0)]. \end{aligned}$$

Noting that as $\bar{u} > 0$, $\bar{u} > u_0 = S(u_0)$ and $\|\bar{u}\|_\infty = o(\|u\|_\infty)$, $\|u_0\|_\infty = o(\|u\|_\infty)$, it is easy to see that $\|A - B\| = o(\|u\|_\infty)$. Therefore $\|S(u_0 + u) - S(u_0) - S_L(u)\| = \|\bar{u} - u_0 - v\| = o(\|u\|_\infty)$. This completes the proof. \square

LEMMA 2.4. *Suppose $a \in C^1(\bar{\Omega})$, $b \in L^\infty(\Omega)$. Then there exists $u > 0 \in C^2(\bar{\Omega})$ and a unique λ_1 such that*

$$\begin{cases} a(x)\Delta u + b(x)u = \lambda_1 u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, λ_1 is increasing in $a(x)$ and in the ratio $b(x)/a(x)$.

LEMMA 2.5. *The only solution to the linearized problem*

$$\begin{cases} -\varphi(x, u_0)\Delta w = w[f(x, u_0) + u_0 f_u(x, u_0)] \\ w = 0 \end{cases} \quad \text{on } \partial\Omega$$

where u_0 is a unique solution to the equation

$$(2.1) \quad \begin{cases} -\varphi(x, u)\Delta u = u f(x, u) \\ u = 0 \end{cases} \quad \text{on } \partial\Omega$$

is $w = 0$.

PROOF. First observe that $\lambda_1(\varphi(x, u_0)\Delta + f(x, u_0)I) = 0$ since $u_0 > 0$ and u_0 is a solution. Since $f(x, u)$ is strictly decreasing in u and $u_0 > 0$, we have $f(x, u_0) + u_0 f_u(x, u_0) < f(x, u_0)$. Therefore using Lemma 2.4, we have

$$\lambda_1[\varphi(x, u_0)\Delta + [f(x, u_0) + u_0 f_u(x, u_0)]I] < \lambda_1[\varphi(x, u_0)\Delta + f(x, u_0)I] = 0$$

Thus $w \equiv 0$ by using the maximum principle (Proposition 2.2) to imply the uniqueness of solutions to equation (2.1). \square

The following two lemmas can be found in Amann [3].

Let (E, P) be an arbitrary ordered Banach space with its usual positive cone P .

LEMMA 2.6. *Let $f : \overline{P}_\rho \rightarrow P$ be a compact map, where $P_\rho = B_\rho(0) \cap P$, $\rho > 0$. If $f(x) \neq \lambda x$ for any $x \in S_\rho^+ := (\partial B_\rho(0)) \cap P$ and every $\lambda \geq 1$, then $\text{index}_P(f, P_\rho) = 1$.*

LEMMA 2.7. *Let $f : \overline{P}_\rho \rightarrow P$ be a compact map such that $f(0) = 0$. Suppose that f has a right derivative $f'_+(0)$ at zero such that 1 is not an eigenvalue of $f'_+(0)$ corresponding to a positive eigenfunction. Then there exists a constant $\sigma_0 \in (0, \rho]$ such that for every $\sigma \in (0, \sigma_0]$, $\text{index}_P(f, P_\sigma) = 0$ if $f'_+(0)$ has a positive eigenfunction corresponding to an eigenvalue greater than one.*

Let $T : E \rightarrow E$ be a linear operator on a Banach space. Denote the spectral radius of T by $r(T)$.

LEMMA 2.8. *Assume that T is a compact positive linear operator on an ordered Banach space. Let $u > 0$ be a positive element. Then*

- (i) *If $Tu > u$, then $r(T) > 1$.*
- (ii) *If $Tu < u$, then $r(T) < 1$.*
- (iii) *If $Tu = u$, then $r(T) = 1$.*

3. Proof of Theorem 1.2

It is not hard to see that using strong maximum principle, if $(u(x), v(x))$ is a positive solution of the system (1.1), then

$$0 < u(x) < u_0(x) < C_1, \quad 0 < v(x) < v_0(x) < C_2.$$

(See the proof of Lemma 5 in [2].)

We will prove the result for the case which the first eigenvalues of two operators in Theorem 1.2 (ii) are both negative. One can refer to [1] for the other case.

By continuity of the functions f, g, u, v on a compact set $\bar{\Omega}$, we can find $M > 0$ large enough that

$$\max\{\max|f(u(x), v(x))|, \max|g(u(x), v(x))|\} < M$$

Define operator :

$$\begin{aligned} A(u, v) \\ := [(-\varphi(\cdot, v)\Delta + M)^{-1}[uf(u, v) + Mu], (-\psi(u, \cdot)\Delta + M)^{-1}[vg(u, v) + Mv]] \end{aligned}$$

Then A is the direct sum of positive compact operator. Note that system has a solution (u, v) if and only if (u, v) is a fixed point of A .

We introduce the following notations.

$$D := \{(u, v) \in C_0(\Omega) \oplus C_0(\Omega) \mid u \leq C_1 + 1, v \leq C_2 + 1\}$$

$$K := \{u \in C_0(\Omega) \mid 0 \leq u(x), x \in \bar{\Omega}\}$$

$$W := K \oplus K$$

$$P_\rho := \{(u, v) \in W \mid u \leq \rho, v \leq \rho\}, \quad \rho > 0$$

$$D' := (\text{int}D) \cap (K \oplus K)$$

Note that D' is open in W . Now we will prove Theorem 1.2 by the sequence of lemmas.

LEMMA 3.1. *Assume $\lambda_1(\varphi(0, 0)\Delta + f(0, 0)) > 0$ and $\lambda_1(\psi(0, 0)\Delta + g(0, 0)) > 0$. Then*

$$\text{index}(A, D', K \oplus K) = 1$$

PROOF. Let $\rho = \max\{C_1, C_2\} + 1$. Then it is easy to see that A has no fixed points on the boundary of P_ρ , ∂P_ρ . So $\lambda_1 = 1$ is not an eigenvalue of A with eigenvector on ∂P_ρ .

Suppose that there is a pair $(\phi_1, \phi_2) \in \partial P_\rho$ for some $\lambda > 1$ such that $A(\phi_1, \phi_2) = \lambda(\phi_1, \phi_2)$. Then we have

$$\varphi(\lambda\phi_1, \phi_2)\Delta\phi_1 + \phi_1 f(\phi_1, \phi_2)/\lambda = (M - M/\lambda)\phi_1$$

$$\psi(\phi_1, \lambda\phi_2)\Delta\phi_2 + \phi_2 g(\phi_1, \phi_2)/\lambda = (M - M/\lambda)\phi_2$$

If ϕ_1 attains its maximum at x_0 , i.e., $\phi_1(x_0) = \rho$, then $\Delta\phi_1(x_0) < 0$ and $\varphi(\lambda\phi_1(x_0)) > 0$. Thus it follows that $f(\phi_1(x_0), \phi_2(x_0)) > 0$. However, by the fact $\phi_1(x_0) > C_1$ and assumptions (H1)-(H2),

$$f(\phi_1(x_0), \phi_2(x_0)) < f(\phi_1(x_0), 0) < f(C_1, 0) = 0,$$

which is a contradiction. Therefore $\lambda > 1$ can not be an eigenvalue of A with eigenvector $(\phi_1, \phi_2) \in \partial P_\rho$. Thus by Lemma 2.6, we have that $\text{index}_W(A, P_\rho) = 1$. \square

LEMMA 3.2. Assume that $\lambda_1(\varphi(0, 0)\Delta + f(0, 0)) > 0$ and $\lambda_1(\psi(0, 0)\Delta + g(0, 0)) > 0$. Then

$$\text{index}_W(A, (0, 0)) = 0.$$

PROOF. We have $A(0, 0) = (0, 0)$ and A is compact on P_ρ . We introduce the notations for the simplicity. Let

$$H(u, v) := (-\varphi(u, v)\Delta + M)^{-1}$$

$$R(u, v) := (-\psi(u, v)\Delta + M)^{-1}.$$

Set

$$L := A'(0, 0) = \begin{bmatrix} H(0, 0)[f(0, 0) + M] & 0 \\ 0 & R(0, 0)[g(0, 0) + M] \end{bmatrix}$$

Suppose that 1 is an eigenvalue of L with a positive eigenvector (ϕ_1, ϕ_2) , i.e.,

$$L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Then we have

$$H(0, 0)^{-1}\phi_1 = (f(0, 0) + M)\phi_1$$

$$R(0, 0)^{-1}\phi_2 = (g(0, 0) + M)\phi_2$$

Hence

$$\lambda_1(\varphi(0, 0)\Delta + f(0, 0)) = \lambda_1(\psi(0, 0)\Delta + g(0, 0)) = 0$$

which is a contradiction. Thus 1 is not an eigenvalue of L .

We claim that there exists $\lambda_1 > 1$ and a corresponding positive eigenvalue of L .

Let $\mu := \lambda_1(\varphi(0, 0)\Delta + f(0, 0)) > 0$ and ϕ_1 the corresponding positive eigenfunction. Then $\varphi(0, 0)\Delta\phi_1 + f(0, 0)\phi_1 = \mu\phi_1 > 0$. Hence $H(0, 0)^{-1}\phi_1 < (f(0, 0) + M)\phi_1$. Thus it follows that $T\phi_1 := H(0, 0)(f(0, 0) + M)\phi_1 > \phi_1$. So by Lemma 2.8, $r[H(0, 0)(f(0, 0) + M)] > 1$. Using the Krein-Rutman theorem, we have that $r(T)$ is an eigenvalue of T with positive eigenfunction ϕ_2 . Thus if we consider the pair $(\phi_2, 0)$ and $\lambda = r(T) > 1$, we have an eigenvalue greater than one with a positive eigenfunction. By using Lemma 2.7, we have that there exists $\sigma_0 \in (0, \rho]$ such that $\text{index}_W(A, P_\sigma) = 0$ for any $0 < \sigma < \sigma_0$. On the other hand, since $(0, 0)$ is isolated, there exists $\delta > 0$ such that $(0, 0)$ is the only fixed point of A in P_δ . If we take $\sigma < \min\{\sigma_0, \delta\}$, then $\text{index}_W(A, (0, 0)) = \text{index}_W(A, P_\sigma) = 0$. \square

LEMMA 3.3. Assume that $\lambda_1(\varphi(0, 0)\Delta + f(0, 0)) > 0$ and $\lambda_1(\psi(0, 0)\Delta + g(0, 0)) > 0$. If

$$\begin{aligned}\lambda_1(\varphi(0, v_0)\Delta + f(0, v_0)I) &< 0 \\ \lambda_1(\psi(u_0, 0)\Delta + g(u_0, 0)I) &< 0,\end{aligned}$$

then

$$\text{index}_W(A, (u_0, 0)) = \text{index}_W(A, (0, v_0)) = 1$$

PROOF. We will only calculate $\text{index}_W(A, (u_0, 0))$ since we can argue similarly for the case $\text{index}_W(A, (0, v_0))$.

Recall $\overline{W}_y = C_0(\Omega) \oplus K$ and

$$\begin{aligned}L &:= A'(u_0, 0) \\ &= \begin{bmatrix} H(u_0, 0)[f(u_0, 0) + u_0 f_u(u_0, 0) + M] & R(u_0, 0)[u_0 f_v(u_0, 0)] \\ 0 & R(u_0, 0)[g(u_0, 0) + M] \end{bmatrix}\end{aligned}$$

First we show that $I - L$ is invertible on $C_0(\Omega) \oplus C_0(\Omega)$.

Suppose that there exists functions $\phi_1, \phi_2 \in C_0(\Omega)$ such that

$$L \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

i.e.,

$$\begin{aligned}H(u_0, 0)[f(u_0, 0) + u_0 f_u(u_0, 0) + M]\phi_1 + R(u_0, 0)[u_0 f_v(u_0, 0)]\phi_2 &= \phi_1 \\ R(u_0, 0)[g(u_0, 0) + M]\phi_2 &= \phi_2\end{aligned}$$

Then it implies that

$$(3.1) \quad \varphi(u_0, 0)\Delta\phi_1 + [f(u_0, 0)u_0f_u(u_0, 0)]\phi_1 = -u_0f_v(u_0, 0)\phi_2$$

and

$$(3.2) \quad \psi(u_0, 0)\Delta\phi_2 + g(u_0, 0)\phi_2 = 0$$

$$(\phi_1, \phi_2) = (0, 0) \text{ on } \partial\Omega.$$

Suppose $\phi_2 \neq 0$. Then 0 is an eigenvalue of $\psi(u_0, 0)\Delta + g(u_0, 0)I$ from (3.2), which is a contradiction due to $\lambda_1(\psi(u_0, 0)\Delta + g(u_0, 0)I) < 0$. Thus $\phi_2 \equiv 0$. So (3.1) becomes

$$\begin{cases} \varphi(u_0, 0)\Delta\phi_1 + [f(u_0, 0) + u_0f_u(u_0, 0)]\phi_1 = 0 \\ \phi_1 = 0 \end{cases} \quad \text{on } \partial\Omega.$$

Then from Lemma 2.5, we have that $\phi_1 \equiv 0$. Therefore $(\phi_1, \phi_2) \equiv (0, 0)$ and $I - L$ is invertible on $C_0(\Omega) \oplus C_0(\Omega)$.

Next we show that L does not have *property* α in \bar{W}_y . Recalling

$$S_y = C_0(\Omega) \oplus \{0\}$$

$$\bar{W}_y \setminus S_y = C_0(\Omega) \oplus \{K \setminus \{0\}\},$$

we suppose L has *property* α in \bar{W}_y . Then there exists a $0 < t < 1$ and functions $(\phi_1, \phi_2) \in \bar{W}_y \setminus S_y$ such that

$$(I - tL) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in S_y,$$

i.e.,

$$(3.3) \quad \phi_1 - t[H(u_0, 0)(f(u_0, 0) + u_0f_u(u_0, 0) + M)\phi_1 \\ + H(u_0, 0)[u_0f_v(u_0, 0)]\phi_2] \in C_0(\Omega)$$

$$(3.4) \quad \phi_2 - tR(u_0, 0)[g(u_0, 0) + M]\phi_2 = 0$$

Note that equation (3.3) holds for arbitrary ϕ_1, ϕ_2 and from equation (3.4), using the fact $\phi_2 \in K \setminus \{0\}$, we have if $T := R(u_0, 0)(g(u_0, 0) + M)I$, then $(I - tT)\phi_2 = 0$. So $T\phi_2 = \phi_2/t > \phi_2$. Thus $r(T) > 1$ by Lemma 2.8. On the other hand, using the assumption $\lambda_1(\psi(u_0, 0)\Delta + g(u_0, 0)I) < 0$, one can show that $r(T) < 1$, which is a contradiction. Hence L does not have *property* α in \bar{W}_y . Thus by Proposition 2.1 (ii) we conclude that

$$\text{index}_W(A, (u_0, 0)) = \text{index}_E(L, (0, 0)) = \pm 1$$

where $E = C_0(\Omega) \oplus C_0(\Omega)$.

Now we calculate $\text{index}_E(L, (0, 0))$ by using the formula

$$\text{index}_E(L, (0, 0)) = (-1)^\sigma$$

where σ is the sum of the multiplicities of the eigenvalues of $L > 1$. Suppose that λ is an eigenvalue of L with eigenvector (ϕ_1, ϕ_2) . Then we have

$$H(u_0, 0)[f(u_0, 0) + u_0 f_u(u_0, 0) + M]\phi_1 + u_0 f_v(u_0, 0)\phi_2 = \lambda\phi_1$$

$$R(u_0, 0)[g(u_0, 0) + M]\phi_2 = \lambda\phi_2.$$

Then $T\phi_2 = \lambda\phi_2$ where T is as above. Since $r(T) < 1$, we have $\lambda < 1$. So there is no eigenvalues of L greater than 1. Hence $\sigma = 0$ and $\text{index}_W(A, (u_0, 0)) = 1$. \square

PROOF OF MAIN THEOREM 1.2. By Lemma 3.1, we have $\text{index}(A, D', K \oplus K) = 1$. To prove that system has a strictly positive solution (u, v) , we will show that A has a nontrivial fixed point in D' . So we need to calculate the fixed-point index for the trivial solution (u, v) and semi-trivial solutions $(u_0, 0)$ and $(0, v_0)$. We also require that the point be an isolated fixed point to use the fixed-point index for an operator at a point. Since we consider the operator A on the set D' if these fixed points are not isolated, then there must be a nontrivial fixed point in the interior of D' . So system has a positive solution. Therefore we may assume that $(0, 0)$, $(u_0, 0)$ and $(0, v_0)$ are isolated fixed point of A . By Lemma 3.2 and Lemma 3.3, we have

$$\text{index}_W(A, (0, 0)) = 0$$

$$\text{index}_W(A, (u_0, 0)) = 1$$

$$\text{index}_W(A, (0, v_0)) = 1$$

By using the excision and solution properties for the index theory, we conclude that A has a nontrivial fixed point in D' . Therefore the system (1.1) has a strictly positive solution. \square

REMARK. One can prove the result that the system (1.1) has a positive solution if the sign of the first eigenvalues of operators $\varphi(0, v_0)\Delta + f(0, v_0)$ and $\psi(u_0, 0)\Delta + g(u_0, 0)$ are both positive by fixed-point index theory used in this paper. It will give the alternative proof for the existence theorem in [2].

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