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F-RATIONALITY OF A PURE SUBRING OF AN F-RATIONAL RING

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ABSTRACT. In this paper we will show that the pure subring R of an F-rational ring S is F-rational when R is a one-dimensional ring, or S is a Gorenstein ring. And we will give a condition that a pure subring of an F-rational ring is to be F-rational.

1. Introduction

M. Hochster and C. Huneke introduced the notions of the tight closure of an ideal and of the weak F-regularity of a ring, and more. Tight closure theory has produced a host of new results and improvements of old results. The applications include invariant theory, the Briançon-Skoda theorem and improved version of the so-called "local homological conjectures" (these conjectures have been proved, for the most part). However, recently M. Hochster gave a list of selected open questions related to the tight closure theory in his paper "Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure [3]." In this paper we will study about the following question in Hochster's list.

"Is a pure subring of an F-rational ring F-rational?"

In fact, we can prove that the above question is affirmative in special cases.

2. Main Theorem

All rings are commutative, with identity, and Noetherian of positive prime characteristic p, unless otherwise specified.

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DEFINITION 2.1 [HOCHSTER-HUNEKE]. Let I be an ideal of R and R^o denote the complement of the union of the minimal prime ideals of R. We say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^o$ such that for all $e \gg 0$, $cx^q \in I^{[q]}$ where $I^{[q]} = (a^q : a \in I)$ when $q = p^e$. If $I = I^*$, then I is called *tightly closed* . R is called *weakly F-regular* if all ideals are tightly closed and R is *F-regular* if all of its localizations are weakly F-regular.

In [1], R. Fedder and K. Watanabe define the weak notion of the weak F-regularity. Here we recall that an ideal I is called a *parameter ideal* for every prime p containing I, IR_p is generated by part of a system of parameters for local ring R_p.

DEFINITION 2.2 [FEDDER-WATANABE]. A ring is called *F-rational* if all parameter ideals are tightly closed.

It is known that if some parameter ideal in a Cohen-Macaulay local ring R is tightly closed, then R is F-rational [1].

In [2], M. Hochster and C. Huneke proved the following:

THEOREM 2.3. *Let $R \subseteq S$ be noetherian rings of characteristic p such that every ideal of R is contracted from S and $R^o \subseteq S^o$. If S is F-regular or weakly F-regular, then R has the same property.*

PROOF. See Proposition 4.11 of [2]. □

But we don't know whether the theorem still holds when weak F-regularity is replaced by F-rationality. In fact, Hochster suggested the following question in his paper [3].

QUESTION [3]. *Is a pure subring R of an F-rational ring S F-rational?*

We can answer this question in the case that the dim$R$=1 or S is a Gorenstein ring.

LEMMA 2.4. *Let R, S be Noetherian rings of characteristic p. Suppose h is a homomorphism from R to S. If $h(R^o) \subseteq S^o$ (which is equivalent to the assertion that every minimal prime of S contracts to a minimal prime of R), then we have the following:*

(a) *If $I \subseteq R$, then $h(I^*) \subseteq (IS)^*$*

(b) *If J is tightly closed in S, its contraction to R is tightly closed in R.*

PROOF. To see (a), note that $c \in R^o$ and $cx^q \in I^{[q]}$ for all $q \gg 0$ implies that $h(c) \in S^o$ and $h(c)h(x)^q \in (IS)^{[q]}$ for all $q \gg 0$. (b) is immediate from (a). □

THEOREM 2.5. *Let R be a pure subring of a module-finite F-rational ring S. If either dim R=1, or S is Gorenstein, then R is also F-rational.*

PROOF. We may assume that R is a normal local domain by the purity [4, Lemma 6.2 and Proposition 6.15]. First, we may assume that dim R=1. Let x be a nonzero element in R. Then xS is also nonzero in S and so xS is a principal ideal of height one. Then xS is tightly closed by the F-rationality of S. Thus xR is also tightly closed by Lemma 2.4. Hence R is F-rational in this case. If S is Gorenstein, then every ideal of S is tightly closed. Thus every parameter ideal of R is tightly closed by Lemma 2.4. So R is also F-rational. □

But we need some additional conditions that Theorem 2.5 still holds for the general case.

THEOREM 2.6. *Let the local ring (R, m) be pure in the module-finite overring (S, M). If the unique maximal ideal M of S is minimal over mS, and S is F-rational, then R is also F-rational.*

PROOF. We may assume that R is a domain [4, Proposition 6.15]. Then $R^o \subseteq S^o$ holds in this case. Let I be an ideal generated by any system of parameters for R. It suffices to show that $I = I^*$. Since M is minimal over mS and S is module-finite over R, IS is a parameter ideal for S. Then IS is tightly closed in S by the F-rationality of S. Since R is S-pure, every ideal is contracted from S by Lemma 2.4. So $(IS)^* \cap R = I$. But

$$(IS)^* \cap R \supset I^*S \cap R = I^*.$$

Hence $I = I^*$ and R is F-rational. □

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