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A NOTE ON CONVERTIBLE $\{0,1\}$ MATRICES

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ABSTRACT. A square matrix A with $perA \neq 0$ is called convertible if there exists a $\{1,-1\}$ matrix H such that $perA = det(H \circ A)$ where $H \circ A$ denote the Hadamard product of H and A. In this paper, ranks of convertible $\{0,1\}$ matrices are investigated and the existence of maximal convertible matrices with its rank r for each integer r with $\lceil \frac{n}{2} \rceil \leq r \leq n$ is proved.

1. Introduction

Let $A = [a_{ij}]$ be any real matrix of order n. The permanent of A is defined by

$$perA = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where S_n denotes the set of permutations of $1, 2, 3, \cdots, n$. An $n \times n$ matrix A with $perA \neq 0$ is called *convertible* if there exists a $\{1,-1\}$ matrix H such that $perA = det(H \circ A)$ where $H \circ A$ denotes the Hadamard product of H and A. In this case, $H \circ A$ is called a conversion of A. A square convertible $\{0,1\}$ matrix is called *maximal* if replacing any zero entry with a 1 results in a non-convertible matrix.

For matrices A, B of the same size, A is said to be *permutation equivalent* to B, denoted by $A \sim B$, if there are permutation matrices P, Q such that $PAQ = B$. If both A and B are real, we use $A \leq B$ to denote that every entry of A is less than or equal to the corresponding entry of B. An $n \times n$ matrix is called *partly decomposable* if it contains a $t \times (n-t)$ zero submatrix for some $t > 0$. Square matrices which are not partly decomposable are called *fully indecomposable.*

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Let $T_n = [t_{ij}]$ denote the $\{0,1\}$ matrix of size $n \times n$ with $t_{ij} = 0$ if and only if $j > i+1$. For a matrix A, square or not, let $\pi(A)$ denote the number of positive entries of A. In [2], it was shown that for any $n \times n$ convertible $\{0,1\}$ matrix A with $perA > 0$, $\pi(A) \leq \pi(T_n) = (n^2+3n-2)/2$ with equality if and only if $A \sim T_n$. In [3,4,5 and 6], the authors investigated some properties of maximal convertible matrices. In this paper, ranks of convertible $\{0,1\}$ matrices are investigated and the existence of maximal convertible matrices with its rank r for each integer r with $\lceil \frac{n}{2} \rceil \leq r \leq n$ is proved. For an $n \times n$ matrix A and for $\alpha,\ \beta \subset \{1,2,\cdots,n\}$, let $A(\alpha|\beta)$ denote the submatrix obtained from A by deleting rows α and columns β and let $A[\alpha|\beta]$ denote the matrix complementary to $A(\alpha|\beta)$ in A. Let $J_{n,m}$ denote the $n \times m$ matrix all of whose entries are 1 and let E_{ij} denote the $n \times n$ matrix all of whose entries are 0 except for the (i,j) entry which is 1.

2. Ranks of Convertible {0,1} Matrices

Let $U_2 = T_2$ and let

$$U_n = \begin{pmatrix} 1 & \mathbf{a} \\ \mathbf{b} & U_{n-1} \end{pmatrix}$$

for $n \geq 3$ where

$$\mathbf{a} = (1, \frac{1+(-1)^n}{2}, 0, \cdots, 0),\ \mathbf{b} = (1, \frac{1-(-1)^n}{2}, 0, \cdots, 0)^T.$$

Then U_n is convertible and it is easy to show that the rank of U_n is $\lceil \frac{n}{2} \rceil$ where $\lceil x \rceil$ denotes the smallest integer not less than x. Hence the minimum rank of $n \times n$ convertible $\{0,1\}$ matrices is not more than $\lceil \frac{n}{2} \rceil$. If $\{0,1\}$ matrix A of size $n \times n$ with $perA > 0$ is of rank 1, then $A = J_{n,n}$. However $J_{n,n}$ is not convertible for $n \geq 3$. Thus we ask a question about the minimum rank of square convertible $\{0,1\}$ matrices with positive permanents. Let $r(A)$ denote the rank of a matrix A.

PROBLEM. *If A is a $n \times n$ convertible $\{0,1\}$ matrix with $perA > 0$, then $r(A) \geq \lceil \frac{n}{2} \rceil$?*

Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be a convertible $\{0,1\}$ matrix of order n. Then for $k \in \{1, 2, \cdots, n\}$,

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{a}_k & \mathbf{a}_1 & \cdots & \mathbf{a}_{k-1} & \mathbf{a}_k & \mathbf{a}_{k+1} & \cdots & \mathbf{a}_n \end{pmatrix}$$

is a convertible matrix of order $n+1$. A convertible matrix C is called a column expansion of convertible matirx A if $C = PBQ$ for some permutation matrices P, Q. A row expansion of a convertible matrix is similarly defined. A matrix is called an *expansion* of convertible matrix A if it is a row expansion or a column expansion of A. Let $\mathcal{A}_n$ be the set of all $n \times n$ convertible $\{0,1\}$ matrices A with the minimum rank and $perA > 0$.

THEOREM 2.1. *Let $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_{n+1}$. Then $r(B) = r(A)$ or $r(B) = r(A) + 1$.*

PROOF. Since B is a $\{0,1\}$ matrix with $perB > 0$, we may assume that

$$B = \begin{pmatrix} 1 & * \\ * & C \end{pmatrix}$$

where $perC > 0$. Then C is a $n \times n$ convertible matrix. Hence $r(B) \geq r(C) \geq r(A)$. Consider an expansion A^e of A. Then A^e is also $(n+1) \times (n+1)$ convertible matrix. Thus $r(A^e) \geq r(B)$ but $r(A^e) = r(A)$ or $r(A^e) = r(A) + 1$. Hence $r(B) = r(A)$ or $r(B) = r(A) + 1$. □

Two vectors $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n)$ are said to have *same zero patterns* if $x_i = 0$ implies $y_i = 0$, and vice versa. Otherwise the two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to have *different zero patterns.*

LEMMA 2.2. *Let A be an $n \times n$ $\{0,1\}$ matrix with $perA > 0$. If the number of rows of different zero patterns in A is less than $\lceil \frac{n}{2} \rceil$, then A is not convertible.*

PROOF. Suppose that A is convertible. Let $A^T = [\mathbf{a}_1^T, \cdots, \mathbf{a}_n^T]$. Without loss of generality, we may assume that $\mathbf{a}_1, \cdots, \mathbf{a}_r$ are the rows of different zero patterns in A. Then $r(A) \leq r < \lceil \frac{n}{2} \rceil$. This implies that A has at least three identical rows, say, $\mathbf{a}_1, \mathbf{a}_s, \mathbf{a}_t$. Since $perA > 0$, there exists $\sigma \in S_n$ such that $\prod_{i=1}^{n} a_{i\sigma(i)} > 0$ and the number of nonzero entries in $\mathbf{a}_1$ is not less than 3. Since $perA(1, s, t|\sigma(1), \sigma(s), \sigma(t)) > 0$, $A[1, s, t|\sigma(1), \sigma(s), \sigma(t)] = J_3$ is convertible, which is impossible. □

Two vectors $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ are said to be *disjoint* if $x_i \neq 0$ implies $y_i = 0$ and vice versa for all $i = 1, 2, \cdots, n$.

THEOREM 2.3. *Let A be an $n \times n$ $\{0,1\}$ matrix having identity matrix I_k of order k as a submatrix and $perA > 0$. If A is convertible with $r(A) = k$, then $k \geq \lceil \frac{n}{2} \rceil$.*

PROOF. Without loss of generality, we may assume that A is of the form

$$A = \begin{pmatrix} I_k & B \\ C_1 & C_2 \end{pmatrix}.$$

Suppose that that $k < \lceil \frac{n}{2} \rceil$. Since $r(A) = k$, any row of $C = [C_1, C_2]$ is a linear combination of the first k rows of A. Since the first k rows of A contains I_k, any row of C is a linear combination of the first k rows of A such that each component scalar is 1. That is, $\mathbf{a}_i = \mathbf{a}_{i_1} + \mathbf{a}_{i_2} + \cdots + \mathbf{a}_{i_p}$ where $1 \leq i_1 < i_2 < \cdots < i_p \leq k$ for all $i = k+1, \cdots, n$. Since A is $\{0,1\}$ matrix, the corresponding rows $\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_p}$ are disjoint. Let D be an $n \times n$ $\{0,1\}$ matrix such that $D[1, 2, \cdots, k|1, 2, \cdots, n] = A[1, 2, \cdots, k|1, 2, \cdots, n]$ and choose i-th row $\mathbf{d}_i$ of D as one of $\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_p}$ for all $i = k+1, \cdots, n$ such that $perD > 0$. Then $D \leq A$ and the number of rows of different zero patterns in D is $k < \lceil \frac{n}{2} \rceil$. By Lemma 2.2, D is not convertible. Hence A is not convertible. □

Let $P_n = [p_{ij}]$ be the permutation matrix of order n such that $p_{ij} = 1$ if and only if $(i, j) \in \{(1, 2), (2, 3), \cdots, (n-1, n), (n, 1)\}$. Recall that an $n \times n$ nonnegative matrix A is *doubly indecomposable* if $perA(i, j|k, l) > 0$ for all i, j, k and l.

LEMMA 2.4. *For $n \geq 3$, $W_n = \begin{pmatrix} J_{n-1,1} & I_{n-1} + P_{n-1} \\ 0 & J_{1,n-1} \end{pmatrix}$ is a maximal convertible matrix and $perW_n = (n-1)^2$.*

PROOF. Let

$$H = \begin{cases} J_{n,n} - 2(\sum_{k=1}^{n/2-1} E_{2k,1} + \sum_{k=2}^{n/2} E_{n,2k} + E_{1,2}) & \text{if } n \text{ is even} \\ J_{n,n} - 2(\sum_{k=1}^{(n-1)/2} E_{2k,1} + \sum_{k=1}^{(n-1)/2} E_{n,2k}) & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to prove that $perW_n = det(H \circ W_n) = (n-1)^2$. Maximality of W_n comes from the fact that a doubly indecomposable convertible $\{0,1\}$ matrix doesn't have a $J_{2,3}$ or $J_{3,2}$ as its submatrix. □

Notice that in Lemma 2.4,

$$det(A) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

For $\{0,1\}$ matrices

$$A = \begin{pmatrix} A_1 & \mathbf{a}_2 \\ \mathbf{a}_1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & \mathbf{b}_2 \\ \mathbf{b}_1 & B_1 \end{pmatrix},$$

let

$$A \star B = \begin{pmatrix} A_1 & \mathbf{a}_2 & \mathbf{O} \\ \mathbf{a}_1 & 1 & \mathbf{b}_2 \\ \mathbf{b}_1\mathbf{a}_1 & \mathbf{b}_1 & B_1 \end{pmatrix}.$$

LEMMA 2.5. *Let A and B be maximal convertible matrices. Then $A \star B$ is a maximal convertible matrix.*

PROOF. Let the sizes of A and B be $k \times k$ and $l \times l$ respectively.

Expanding the permanent of $A \star B = C = [c_{ij}]$ by k-th row, we have

$$\begin{aligned} perC &= \sum_{j=1}^{k+l-1} c_{kj} perC(k|j) = \sum_{j=1}^{k} c_{kj} perC(k|j) \\ &+ \sum_{j=k+1}^{k+l-1} c_{kj} perC(k|j) = \sum_{j=1}^{k} a_{kj} perA(k|j) perB(1|1) \\ &+ perA \sum_{j=k+1}^{k+l-1} c_{kj} \sum_{p=k+1}^{k+l-1} c_{pk} perB(1, p-k+1|1, j-k+1) \\ &= perA\{perB(1|1) + \sum_{j=2}^{l} b_{1j} \sum_{p=2}^{l} b_{p1} perB(1,p|1,j)\} \\ &= perA\{perB(1|1) + \sum_{j=2}^{l} b_{1j} perB(1|j)\} \\ &= perA \cdot perB. \end{aligned}$$

Let $H = [h_{ij}]$ and $K = [k_{ij}]$ be converters of A and B with $h_{kk} = k_{11} = k_{21} = \cdots = k_{l1} = 1$ respectively and let $L = H \star K$. We write $A^* = [a^*_{ij}] = H \circ A$, $B^* = [b^*_{ij}] = K \circ B$ and $C^* = [c^*_{ij}] = L \circ C$.

For j with $k+1 \le j \le k+l-1$, expanding $detC^*(k|j)$ by the first k columns, we have

$$\begin{aligned} detC^*(k|j) &= detA^* \sum_{p=k+1}^{k+l-1} (-1)^{k+p} c^*_{pk} detB^*(1, p-k+1|1, j-k+1) \\ &= detA^* \sum_{s=2}^{l} (-1)^{s+1} b^*_{s1} detB^*(1, s|1, j-k+1) \\ &= detA^* \cdot detB^*(1|j-k+1). \end{aligned}$$

Thus

$$\begin{aligned} detC^* &= \sum_{j=1}^{k+l}(-1)^{k+j}c^*_{kj}detC^*(k|j) = \sum_{j=1}^{k}(-1)^{k+j}c^*_{kj}detC^*(k|j) \\ &+ \sum_{j=k+1}^{k+l-1}(-1)^{k+j}c^*_{kj}detC^*(k|j) = \sum_{j=1}^{k}(-1)^{k+j}a^*_{kj}detA^*(k|j)detB^*(1|1) \\ &+ \sum_{j=k+1}^{k+l-1}(-1)^{k+j}c^*_{kj}(detA^* \cdot detB^*(1|j-k+1)) \\ &= detA^*(detB^*(1|1) + \sum_{t=2}^{l}(-1)^{t+1}b^*_{1t}detB^*(1|t)) \\ &= detA^* \cdot detB^* = perA \cdot perB = perC. \end{aligned}$$

Hence C is a convertible matrix. To prove the maximality of C, it is sufficient to show that $C+E_{ij}$ is not convertible for $1 \le i < k$, $k < j \le n$ since A and B are maximal convertible matrices. Suppose that $C + E_{ij}$ is convertible for some i ,j with $1 \le i < k$, $k < j \le n$. Without loss of generality, we may assume that $i = 1$ and $j = n$. Since C is fully indecomposable, $perC(1|n) > 0$. Hence there exists $\sigma \in S_{n-1}$ such that $c_{2\sigma(2)}c_{3\sigma(3)} \cdots c_{n\sigma(n)} = 1$ and $C[i, j|\sigma(i), \sigma(j)] = J_2$ for some i, j with $k \le i$, $j \le n$ and $1 \le \sigma(i)$, $\sigma(j) \le k$. Also we have a converter L' of $C+E_{1n}$ satisfying $L'[i, j|\sigma(i), \sigma(j)] = J_2$. This means $L'[i, j|\sigma(i), \sigma(j)] = J_2$ is a converter of the convertible matrix $C[i, j|\sigma(i), \sigma(j)] = J_2$, which is impossible. Hence $C + E_{1n}$ is not convertible. □

THEOREM 2.6. *There exists a maximal convertible matrix A such that $r(A) = n$ if $n \ge 4 (n \ne 5)$.*

PROOF. If n is even, take the matrix W_n in Lemma 2.4. Then $r(W_n) = n$. If n is odd, let $A = W_{n-3} * (J_{4,4} - \sum_{i=1}^{4} E_{i,5-i})$. Then A is a maximal convertible matrix by Lemma 2.5 and

$$\begin{aligned} det(A) &= det(W_{n-3})det(\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}) \\ &= (n-4) \cdot (-3) \ne 0. \end{aligned}$$

Hence $r(A) = n$. □

It is easy to show that every 3×3 maximal convertible matrix is permutation equivalent to T_3 whose rank is 2. By using the well-known fact([1]) that $3n \le \pi(A) \le (n^2 + 3n - 2)/2$ for any maximal convertible $n \times n$ matrix A, we can show that every 5×5 maximal convertible matrix which has no two identical rows (or columns) is permutation equivalent to W_5 in Lemma 2.4. In fact $r(W_5) \neq 5$. Hence there are no maximal convertible $n \times n$ matrices A with $r(A) = n$ for $n = 3, 5$.

THEOREM 2.7. *For any integer s with $\lceil \frac{n}{2} \rceil \le s \le n$, there exists an $n \times n$ maximal convertible matrix A such that $r(A) = s$, $n \ge 4 (n \neq 5)$.*

PROOF. Let $T_k = [t_{ij}]$ be the lower Hessenberg matrix of order k, i.e., $t_{ij} = 0$ if and only if $i + j \ge k$. Then $r(T_k) = k - 1$. Inductively define a sequence of maximal convertible matrices $M_k, \cdots, M_n$ as follows: Let $M_k = T_k$ and

$$M_{k+t} = \begin{pmatrix} 1 & \mathbf{a} \\ \mathbf{b} & M_{k+t-1} \end{pmatrix}$$

where $\mathbf{a} = (1, \frac{1-(-1)^t}{2}, 0, \cdots, 0)$ and $\mathbf{b} = (1, \frac{1+(-1)^t}{2}, 0, \cdots, 0)$. Notice that M_n is an $n \times n$ maximal convertible matrix such that $r(M_n) = k - 1 + \lfloor \frac{n-k}{2} \rfloor$. For any integer s with $\lceil \frac{n}{2} \rceil \le s < n$, let $k = 2s - n + 2$ or $k = 2s - n + 3$. Then $r(M_n) = s$. Hence the result comes from this fact and Theorem 2.6. □

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