Comm. Korean Math. Soc. **12** (1997), No. 3, pp. 813–827

# MULTIGRID METHODS FOR THE PURE TRACTION PROBLEM OF LINEAR ELASTICITY: FOSLS FORMULATION

CHANG-OCK LEE

ABSTRACT. Multigrid methods for two first-order system least squares (FOSLS) using bilinear finite elements are developed for the pure traction problem of planar linear elasticity. They are *two-stage* algorithms that first solve for the gradients of displacement, then for the displacement itself. In this paper, concentration is given on solving for the gradients of displacement only. Numerical results show that the convergences are uniform even as the material becomes nearly incompressible. Computations for convergence rates are included.

## 1. Introduction

Let $\Omega$ be a convex polygon or a $C^{1,1}$-domain in $\mathbb{R}^2$ with boundary $\Gamma$. Denote the Lamé constants by $\mu$ and $\lambda$ where $(\mu, \lambda)$ belongs to the range $[\mu_1, \mu_2] \times [\lambda_0, \infty)$, for fixed positive constants $\mu_1$, $\mu_2$ and $\lambda_0$. The pure traction boundary value problem for planar linear elasticity is given in the form:

$$\begin{aligned} -\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \sum_{j=1}^{2} \sigma_{ij}(\mathbf{u}) &= 0 \quad \text{on} \quad \Gamma, \quad 1 \le i \le 2, \end{aligned} \tag{1.1}$$

where the symbols $\Delta$, $\nabla$, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta\mathbf{u}$ is the vector of components

Received January 25, 1997. Revised May 15, 1997.

1991 Mathematics Subject Classification: 65F10, 65N55, 73K10.

Key words and phrases: First-order system least squares, Lamé constants, multigrid method, pure traction problem.

This research was partially supported by the Inha University Research Fund, 1995 and by BSRI-96-1436.

$\Delta u_i$). In addition, $\mathbf{u} = (u_1, u_2)^t$ denotes the displacement, $\mathbf{f} = (f_1, f_2)^t$ a given body force, and $\mathbf{n} = (n_1, n_2)^t$ the unit outerward normal on the boundary. Furthermore, $\sigma_{ij}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u})$ is the *stress*, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ is the *strain*, and $\delta_{ij}$ is the Kronecker delta symbol.

It is well-known that finite element method using piecewise linear ($P$-1) finite elements converges for moderate fixed $\lambda$, and as $\lambda \to \infty$, i.e., the elastic material becomes incompressible, it seems not to converge at all ([1, 16]). In order to overcome this so called the locking phenomenon, several attempts have been made in [2, 7, 12, 15]. These attempts are usually based on *mixed* formulations that lead to discrete equations that are difficult to solve. Recently, in [4] Cai et. al. proposed first-order system least squares (FOSLS) for the Stokes equations and applied it to the pure displacement problem of planar linear elasticity. In addition, in [5] FOSLS approach was developed for the pure traction problem. They obtained $H^1$-ellipticity of the least squares functional and showed that $H^1$-ellipticity is independent of $\lambda$. Since the FOSLS formulation generates a symmetric positive (in)definite system, the multigrid method seems to be the fastest method for the induced linear system. Furthermore $H^1$-ellipticity property guarantees optimal finite element accuracy and multigrid convergence. Also $\lambda$-independency of $H^1$-ellipticity gives uniform convergence of multigrid method with respect to $\lambda$.

In this paper we present V-cycle multigrid methods to solve the linear system arising from the bilinear finite element method for two FOSLS formulations of the pure traction problem. The first one is based on the least squares functional for the pure traction problem in [5] and the second one is based on a modification of the least squares functional for the pure displacement problem in [4]. In both cases, the algorithms are *two-stage*, in which one solves for the displacement flux variable first. The displacement components can then be obtained as solutions of two scalar Poisson equations. We concentrate on the implementation of the first stage and test the multigrid efficiency numerically since there are many good Poisson solvers. Furthermore, we show numerically that the multigrid convergences are uniform with respect to $\lambda$. In the implementation of the multigrid algorithms, we do not impose the compatibility condition to the solution space. Instead, we take the projection right

after each smoothing step of the multigrid algorithms to make solutions satisfy the compatibility condition.

This paper is organized as follows. In Section 2 we explain notations, and spaces on which we would solve the problem. FOSLS is discussed and known results are stated in Section 3. In Section 4, we describe FOSLS based on Stokes approach. In Section 5 we give a multigrid algorithm. In the last section numerical results are presented.

## 2. Notations and Preliminaries

Throughout this paper, the letter $C$ denotes a positive constant independent of the Lamé constants and the mesh parameter $h_k$, which may vary from occurrence to occurrence. We frequently use the term *uniform* in reference to a relation to mean that it holds independent of $\lambda$. Sometimes, we call it $\lambda$-independency.

We use standard notations and definitions for the Sobolev spaces $[H^k(\Omega)]^2$, associated inner products $(\cdot,\cdot)_k$, and respective norms $\|\cdot\|_k$, $k > 0$. The space $[L^2(\Omega)]^2$ is interpreted as $[H^0(\Omega)]^2$, in which case the norm and inner product are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. As usual $H_0^k(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_k$. See [6].

Let RM denote the space of rigid motions, which is defined by

$$\mathrm{RM} := \left\{ \mathbf{u} : \mathbf{u} = \begin{pmatrix} a + bx_2 \\ c - bx_1 \end{pmatrix}, \quad a, b, c \in \mathbb{R} \right\},$$

$\mathrm{RM}^\perp$ its orthogonal complement in $[L^2(\Omega)]^2$, and $\widehat{\mathrm{RM}}$ its orthogonal complement in $[H^1(\Omega)]^2$. It is known that $\mathbf{u} \in \widehat{\mathrm{RM}}$ if and only if

$$\int_\Omega \mathbf{u}\, dx = 0, \tag{2.1.a}$$

$$\int_\Omega \nabla \times \mathbf{u}\, dx = 0, \tag{2.1.b}$$

where $\nabla \times \mathbf{u} := \partial_1 u_2 - \partial_2 u_1$. In order for a solution of (1.1) to exist, $\mathbf{f} \in [L^2(\Omega)]^2$ must satisfy the compatibility condition

$$\int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx = 0 \quad \forall \mathbf{v} \in \mathrm{RM}.$$

Then the pure traction problem (1.1) has a unique solution $\mathbf{u} \in [\widehat{H}^2(\Omega)]^2$ where $[\widehat{H}^k(\Omega)]^2 := [H^k(\Omega)]^2 \cap \widehat{\mathrm{RM}}$ and $[H^k_\perp(\Omega)]^2 := [H^k(\Omega)]^2 \cap \mathrm{RM}^\perp$. See [2, 12] for more details.

We will introduce a new independent variable related to the 4-vector function of gradients of $\mathbf{u}$. We view the original 2-vector functions as column vectors and the new 4-vector function as a block column vector. Thus given

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

the operator $\nabla$ is extended to 2-vectors componentwise:

$$\nabla \mathbf{u} = \begin{pmatrix} \nabla u_1 \\ \nabla u_2 \end{pmatrix}.$$

If $\mathbf{U}_1$ and $\mathbf{U}_2$ are 2-vector functions, then we write the block column vector

$$\mathbf{U} \equiv \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix}.$$

If $D$ is an operator on 2-vector functions (e.g., $D = \nabla\cdot$, $\nabla\times$, or $\mathbf{n}\times$), then its extension to block column vectors is defined by

$$D\mathbf{U} = \begin{pmatrix} D\mathbf{U}_1 \\ D\mathbf{U}_2 \end{pmatrix}.$$

Finally, inner products and norms on column vector functions are defined in the natural componentwise way: $\|\mathbf{U}\|^2 = \sum_{i=1}^2 \|\mathbf{U}_i\|^2$.

Since $\mu$ is bounded, we may set $\mu = 1$ without loss of generality. The pure traction problem (1.1) is rewritten in the compact form:

$$\begin{aligned} -\nabla \cdot (A\nabla \mathbf{u}) &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{n} \cdot (A\nabla \mathbf{u}) &= \mathbf{0} \quad \text{on} \quad \Gamma, \end{aligned} \tag{2.2}$$

where

$$A = \begin{pmatrix} \lambda+2 & 0 & 0 & \lambda \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \lambda & 0 & 0 & \lambda+2 \end{pmatrix}.$$

We introduce the *displacement flux* variable $\mathbf{U} = \nabla\mathbf{u}$, that is,

$$\mathbf{U} = (U_1, U_2, U_3, U_4)^t = (\partial_1 u_1, \partial_2 u_1, \partial_1 u_2, \partial_2 u_2)^t .$$

Since the definition of $\mathbf{U}$ implies that $\nabla \times \mathbf{U} = 0$ in $\Omega$, a system that is equivalent to (2.2) is

$$\begin{aligned} \mathbf{U} - \nabla\mathbf{u} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ -\nabla \cdot A\mathbf{U} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla \times \mathbf{U} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \mathbf{n} \cdot A\mathbf{U} &= \mathbf{0} \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.3}$$

This extended system is well posed and suitable for treatment by FOSLS. However, what is more important in practice is the system that involves $\mathbf{U}$ only:

$$\begin{aligned} -\nabla \cdot A\mathbf{U} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla \times \mathbf{U} &= \mathbf{0} \quad \text{in} \quad \Omega, \\ \mathbf{n} \cdot A\mathbf{U} &= \mathbf{0} \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.4}$$

This reduced system is also well posed and it is better suited to FOSLS treatment, especially in the incompressible limit, i.e., $\lambda \to \infty$. See [5] for more details.

We define a solution space for the primitive variables by

$$\mathcal{W} = \{\mathbf{u} \in [\widehat{H}^1(\Omega)]^2 : \nabla \cdot (A\nabla\mathbf{u}) \in [L^2(\Omega)]^2, \mathbf{n} \cdot (A\nabla\mathbf{u}) = \mathbf{0} \text{ on } \Gamma\} .$$

Since we have posed (2.2) on the space $\mathcal{W}$, (2.1.b) implies that

$$\int_\Omega (U_2 - U_3)\, dx = 0, \quad \text{i.e.,} \quad \mathbf{U} \perp (0, 1, -1, 0)^t . \tag{2.5}$$

We thus define the solution space for the new variables by

$$\mathcal{V} = \{\mathbf{U} \in [H^1(\Omega)]^4 : \mathbf{n} \cdot A\mathbf{U} = \mathbf{0} \text{ on } \Gamma, \mathbf{U} \perp (0, 1, -1, 0)^t\} .$$

## 3. First-Order System Least Squares

The primary objective of this section is to establish a least squares functional based on (2.4) in appropriate Sobolev spaces. To this end, we assume that $\mathbf{f} \in [\widehat{L}^2(\Omega)]^2$. In fact, there is an one-to-one correspondence between $[L^2_\perp(\Omega)]^2$ and $[\widehat{L}^2(\Omega)]^2$ (see [2, 12]). Define

$$G_0(\mathbf{U};\mathbf{f}) = \|\mathbf{f} + \nabla \cdot A\mathbf{U}\|^2 + \|\nabla \times \mathbf{U}\|^2 \quad \text{for } \mathbf{U} \in \mathcal{V}.$$

In [5], uniform boundedness and ellipticity (i.e., equivalence) of the functional $G_0(\mathbf{U};\mathbf{0})$ is established in terms of the functional $M(\mathbf{U})$ defined on $\mathcal{V}$ by

$$M(\mathbf{U}) = \|\mathbf{U}\|_1^2 + \lambda^2 \|\nabla \mathrm{tr}\mathbf{U}\|^2,$$

where the *trace* operator tr is defined by $\mathrm{tr}\mathbf{U} = U_1 + U_4$.

Since the domain $\Omega$ is a convex polygon or a $C^{1,1}$-domain, we have standard $H^2$-regularity results for elasticity equation (2.2) (cf. [8, 9]):

$$\|\mathbf{v}\|_2 \le C\|\nabla \cdot (A\nabla\mathbf{v})\| \quad \forall \mathbf{v} \in \mathcal{W} \cap [H^2(\Omega)]^2. \tag{3.1}$$

In fact, if $\nabla \cdot A\nabla\mathbf{v} \in [L^2(\Omega)]^2$, then $H^2$-regularity result implies that $\mathbf{v} \in [H^2(\Omega)]^2$. Furthermore, for any constant $\rho > 0$, the usual Stokes $H^2$-regularity implies that (cf. [10, 11])

$$\|\rho\mathbf{w}\|_2^2 + \|p\|_1^2 \le C\| - \rho\nabla \cdot \nabla\mathbf{w} + \nabla p\|^2, \tag{3.2}$$

for any $p \in H^1(\Omega)/\mathbb{R}$ and $\mathbf{w} \in [H^1_0(\Omega)]^2 \cap [H^2(\Omega)]^2$ such that $\mathbf{w}$ is divergence free.

The basic aim of FOSLS is to develop a functional whose homogeneous form is equivalent to a product norm composed of individual scalar $L^2$ or $H^1$-like norms. The essential purpose of this construction is to reduce the original problem to a system of easily solved scalar equations whose coupling is weak enough to enable relatively easy solution of the full system. While the theory achieves this basic goal, the quality of the relevant coupling degrades as the material properties tend to the incompressible limit (i.e., as $\lambda \to \infty$). The source of this trouble is the term involving $\lambda$ in the definition of functional $M$. Since the expression $\mathrm{tr}\mathbf{U} = U_1 + U_4$ represents an intimate coupling between $U_1$ and $U_4$, large $\lambda$ implies that

the coupling between these two variables must tend to become dominant in the functional. This difficulty, which causes degrading performance of standard solvers, will be eliminated here by a simple rotation applied to $\mathbf{U}$. The rotation we consider is defined by the matrix

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and the space $\tilde{\mathcal{V}} \equiv Q\mathcal{V} = \{\mathbf{V} = Q\mathbf{U} : \mathbf{U} \in \mathcal{V}\}$. Note that $\mathcal{V} = Q\tilde{\mathcal{V}}$ and that each vector $\mathbf{U} \in \mathcal{V}$ is of the form $\mathbf{U} = Q\mathbf{V}$, $\mathbf{V} \in \tilde{\mathcal{V}}$. Note also that spaces $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are the same up to boundary conditions.

The solution $(\mathbf{U}, \mathbf{u})$ of the extended system (2.3) can be obtained as the solution of the following two-stage algorithm:

*Stage* 1: Let $\mathbf{V} \in \mathcal{V}$ be the unique solution of

$$G_0(Q\mathbf{V}; \mathbf{f}) = \min\{G_0(Q\mathbf{W}; \mathbf{f}) : \mathbf{W} \in \mathcal{V}\} \tag{3.3}$$

and set $\mathbf{U} = Q\mathbf{V}$.

*Stage* 2: Define

$$\mathcal{Z} = \{\mathbf{u} \in [H^1(\Omega)]^2 : \int_\Omega \mathbf{u}\, dx = 0\}$$

and let $\mathbf{u} \in \mathcal{Z}$ be the unique solution of

$$\|\nabla \mathbf{u} - \mathbf{U}\| = \min\{\|\nabla \mathbf{v} - \mathbf{U}\| : \mathbf{v} \in \mathcal{Z}\}.$$

Then we have the following theorem, which is found in [5].

THEOREM 3.1. *When (3.1) and (3.2) hold, we have*

$$\frac{1}{C}(\|\mathbf{V}\|_1^2 + \lambda^2 \|\nabla V_1\|^2) \le G_0(Q\mathbf{V}; \mathbf{0}) \le C(\|\mathbf{V}\|_1^2 + \lambda^2 \|\nabla V_1\|^2) \quad \forall \mathbf{V} \in \mathcal{V}. \tag{3.4}$$

$H^1$-equivalence (3.4) immediately implies that standard finite elements and standard multigrid for minimizing $G_0(Q\mathbf{V}; \mathbf{f})$ will achieve

uniform and optimal $H^1$-approximations to $\mathbf{V}$ (cf. [3]). It is clear that the uniqueness of the solutions of the two minimization problems guarantees that $(Q\mathbf{V}, \mathbf{u})$ is the unique solution of the pure traction problem as expressed in (2.3).

For now, let's concentrate only on finding $\mathbf{V}$ to minimize the functional (3.3). Our minimization problem is

$$\min\{G_0(Q\mathbf{V}; \mathbf{0}) : \mathbf{n} \cdot A\mathbf{V} = 0 \text{ on } \Gamma, \mathbf{V} \perp (0, 1, -1, 0)^t\}\,.$$

In order to make (3.4) be real $H^1$-equivalence without $\lambda$ appearing, we will rescale $V_1$: let the new $V_1$ be the old $V_1$ times $\lambda$. The scaling is defined by the matrix

$$D = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{V} = D^{-1}Q\mathbf{U}$ and

$$B = AQD = \begin{pmatrix} \sqrt{2}(1 + \frac{1}{\lambda}) & 0 & 0 & \sqrt{2} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ \sqrt{2}(1 + \frac{1}{\lambda}) & 0 & 0 & -\sqrt{2} \end{pmatrix}.$$

Then $\nabla \cdot A\mathbf{U}$ becomes $\nabla \cdot AQDD^{-1}Q\mathbf{U} = \nabla \cdot B\mathbf{V}$ and $\nabla \times \mathbf{U}$ becomes $\nabla \times QDD^{-1}Q\mathbf{U} = \nabla \times QD\mathbf{V}$.

Our modified problem is

$$\min\{G_1(\mathbf{V}; \mathbf{0}) : \mathbf{n} \cdot B\mathbf{V} = 0 \text{ on } \Gamma, \mathbf{V} \perp (0, 1, -1, 0)^t\}\,, \tag{3.5}$$

where

$$G_1(\mathbf{V}; \mathbf{f}) = \|\mathbf{f} + \nabla \cdot B\mathbf{V}\|^2 + \|\nabla \times QD\mathbf{V}\|^2 \quad \text{for } \mathbf{V} \in \mathcal{V}\,.$$

## 4. FOSLS based on Stokes approach

Recently in [13], a new FOSLS formulation based on the FOSLS for Stokes equation in [4] was proposed for the pure traction problem.

We describe the new formulation briefly. From previous sections, we know that

$$\nabla \cdot A = \begin{pmatrix} (\lambda+2)\partial_1 & \partial_2 & \partial_2 & \lambda\partial_1 \\ \lambda\partial_2 & \partial_1 & \partial_1 & (\lambda+2)\partial_2 \end{pmatrix}$$

and

$$\nabla\times = \begin{pmatrix} \partial_2 & -\partial_1 & 0 & 0 \\ 0 & 0 & \partial_2 & -\partial_1 \end{pmatrix}.$$

Then, first subtract row 2 of $\nabla\times$ from row 1 of $\nabla \cdot A$. Next, add row 1 of $\nabla\times$ to row 2 of $\nabla \cdot A$. Leaving $\nabla\times$ alone, we obtain

$$\nabla \cdot A_s = \begin{pmatrix} (\lambda+2)\partial_1 & \partial_2 & 0 & (\lambda+1)\partial_1 \\ (\lambda+1)\partial_2 & 0 & \partial_1 & (\lambda+2)\partial_2 \end{pmatrix}$$

and

$$\nabla \cdot B_s = \nabla \cdot A_s QD = \begin{pmatrix} \frac{2\lambda+3}{\sqrt{2}\lambda}\partial_1 & \partial_2 & 0 & \frac{1}{\sqrt{2}}\partial_1 \\ \frac{2\lambda+3}{\sqrt{2}\lambda}\partial_2 & 0 & \partial_1 & -\frac{1}{\sqrt{2}}\partial_2 \end{pmatrix}.$$

Now, our new problem is

$$\min\{G_2(\mathbf{V};\mathbf{0}) : \mathbf{n} \cdot B\mathbf{V} = 0 \text{ on } \Gamma, \mathbf{V} \perp (0,1,-1,0)^t\}, \tag{4.1}$$

where

$$G_2(\mathbf{V};\mathbf{f}) = \|\mathbf{f} + \nabla \cdot B_s\mathbf{V}\|^2 + \|\nabla \times QD\mathbf{V}\|^2 \quad \text{for } \mathbf{V} \in \mathcal{V}.$$

In [13], the convergence rate for V(1,0)-multigrid method for this formulation was analyzed to be about 0.5.

## 5. The Multigrid Algorithm

We now turn to a numerical method for the approximation of the solution $\mathbf{V}$ of (3.5) or (4.1). Subdivide $\Omega$ into a set $\mathcal{T}^k$ of non-overlapping

rectangles such that $\Omega = \cup_{T\in\mathcal{T}^k} T$, and no vertex of one rectangle lies on the edge of another rectangle. $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the edges of the rectangles in $\mathcal{T}^k$. Let $h_k := \max_{T\in\mathcal{T}^k} \operatorname{diam} T$, then $h_k = 2h_{k+1}$. Now let us define the finite element space for our multigrid algorithm MG.

$$\begin{aligned}\mathcal{V}_k := \{\mathbf{U} :& \mathbf{U}|_T \text{ is bilinear for all } T \in \mathcal{T}^k,\\ & \mathbf{U} \text{ is continuous on } \Omega, \mathbf{n}\cdot A\mathbf{U} = \mathbf{0} \text{ on } \Gamma\}.\end{aligned}$$

Note that $\mathcal{V}_{k+1} \subset \mathcal{V}_k \subset [H^1(\Omega)]^4$. Then the following approximation property holds: there exists a constant $C$ such that for all $\mathbf{V} \in \mathcal{V}$, there exists $\mathbf{V}^h \in \mathcal{V}_k$ such that

$$\|\mathbf{V}_j - \mathbf{V}_j^h\| + h_k\|\mathbf{V}_j - \mathbf{V}_j^h\|_1 \le Ch_k^2\|\mathbf{V}_j\|_2, \qquad j = 1,\dots,4.$$

**The k-th level iteration scheme of the multigrid algorithm MG:** The $k$-th level iteration with initial iterate $\mathbf{V}_0 \in \mathcal{V}_k$ yields $\mathrm{MG}(k, \mathbf{V}_0, F)$ as an approximate solution to the following problem:
Find $\mathbf{V} \in \mathcal{V}_k$ satisfying the linear system

$$\mathcal{G}_k(\mathbf{V}) = F \quad (= F(\mathbf{f}, \mathbf{V})). \tag{5.1}$$

Here, the above linear system (5.1) comes from the discretization process of the minimization problems (3.5) or (4.1) on the finite element space $\mathcal{V}_k$.

For $k = 1$, $\mathrm{MG}(1, \mathbf{V}_0, F)$ is the solution obtained from a direct method, i.e.,

$$\mathrm{MG}(1, \mathbf{V}_0, F) = \mathcal{G}_1^{-1}F.$$

For $k > 1$,

(1) Pre-Smoothing Step: the approximation $\mathbf{V}_{m_1} \in \mathcal{V}_k$ is constructed by $m_1$ point Gauss-Seidel iterations with an initial iterate $\mathbf{V}_0$. Here, $m_1$ is an integer to be determined later.

(2) Correction Step: The coarser-grid correction in $\mathcal{V}_{k-1}$ is obtained by applying the $(k-1)$-st level iteration. In other words, it is the standard V-cycle multigrid method. More precisely,

$$\mathbf{W} = \mathrm{MG}(k-1, \mathbf{0}, \bar{F})$$

where $\bar{F}$ is defined by

$$\bar{F} := I_k^{k-1}\left(F - \mathcal{G}_k(\mathbf{V}_{m_1})\right) .$$

Here, $(I_k^{k-1})^t = I_{k-1}^k$ and $I_{k-1}^k$ is bilinear interpolation.

(3) Post-Smoothing Step: the approximation $\mathbf{V}_{m_2} \in \mathcal{V}_k$ is constructed by $m_2$ point Gauss-Seidel iterations with the initial iterate $\mathbf{V}_{m_1} + I_{k-1}^k \mathbf{W}$. Here, $m_2$ is an integer to be determined later as well.

(4) Put

$$\mathrm{MG}(k, \mathbf{V}_0, F) = \mathbf{V}_{m_2} .$$

When we construct the linear system (5.1), we do not impose the condition (2.5) for $k \geq 2$ because the space $\mathcal{V}_k$ has a natural coordinate system which consists of the values of piecewise bilinear functions at mesh points. However, we have to impose (2.5) in the coarsest grid because we need to solve the problem by direct method in the coarsest grid. Therefore we obtain the singular system (5.1) for $k > 1$. To get a unique solution we take Gram-Schmidt process right after each smoothing step of the multigrid algorithm, i.e.,

$$\begin{pmatrix} U_2 \\ U_3 \end{pmatrix} \leftarrow \begin{pmatrix} U_2 \\ U_3 \end{pmatrix} - \frac{\int_\Omega (U_2 - U_3)\, dx}{2 \int_\Omega dx} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

## 6. Experimental Results

For our numerical experiments, we choose a model problem with $\mu = 1$:

$$-\mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} = \mathbf{0} \quad \text{in} \quad \Omega = \text{ unit square},$$

$$\sum_{j=1}^{2} \sigma_{ij}(\mathbf{u}) = 0 \quad \text{on} \quad \Gamma, \quad 1 \leq i \leq 2 .$$

Of course, the exact solution is zero. We take a uniform grid on $\Omega$ and wild initial iterates. The experiments reported here were run in double-precision arithmetic on a SUN Sparc-20 Workstation.

The multigrid algorithms in this section are called one-sided methods and denoted by V(1,0)-MG, i.e., we use one pre-smoothing step and no

TABLE 1. Convergence rate of V(1,0)-MG based on $G_1$

| | $h = \frac{1}{4}$ | $h = \frac{1}{8}$ | $h = \frac{1}{16}$ | $h = \frac{1}{32}$ | $h = \frac{1}{64}$ |
|---|---|---|---|---|---|
| $\lambda = 10$ | 0.5796 | 0.7556 | 0.8065 | 0.8075 | 0.8114 |
| $\lambda = 100$ | 0.5996 | 0.8003 | 0.8274 | 0.8319 | 0.8340 |
| $\lambda = 1000$ | 0.6019 | 0.8028 | 0.8296 | 0.8345 | 0.8365 |

TABLE 2. Convergence rate of V(1,0)-MG based on $G_2$

| | $h = \frac{1}{4}$ | $h = \frac{1}{8}$ | $h = \frac{1}{16}$ | $h = \frac{1}{32}$ | $h = \frac{1}{64}$ |
|---|---|---|---|---|---|
| $\lambda = 10$ | 0.4728 | 0.6030 | 0.6271 | 0.6541 | 0.6554 |
| $\lambda = 100$ | 0.4691 | 0.5822 | 0.6378 | 0.6513 | 0.6577 |
| $\lambda = 1000$ | 0.4688 | 0.5790 | 0.6399 | 0.6520 | 0.6590 |

post-smoothing step. If both smoothing steps are used, the multigrid method is called symmetric and denoted by V(1,1)-MG. Note that as far as the convergence is concerned a V(1,1)-multigrid is the same as two V(1,0)-multigrid methods (See [14]).

We obtained the efficiency of multigrid algorithms by measuring the convergence rate, i.e., ratio of successive square roots of functional values $G_i(\mathbf{V}; 0)$ $(i = 1, 2)$ since the least squares functionals are equivalent to the square of $H^1$-norm.

Table 1 and Table 2 represent convergence rates after 20 V(1,0)-cycles for acceptable convergence. The convergence rates are degrading as $h$ tends to zero, but there seem to be uniform bounds, which coincides with the theory. Usually V(1,0)-multigrid method using $P-1$ finite elements for Poisson equation shows that the convergence rate with respect to the energy norm is about 0.3. Our numerical computations show that the multigrid algorithm using FOSLS for the pure traction problem may not be as good as the one for Poisson equation. We may think that solving the problem for 4-vectors simultaneously causes the poor convergence rates and, in addition, the coupling of boundary conditions also causes the deterioration.

Figure 1 and Figure 2 represent the solutions after 20 V(1,0)-MG

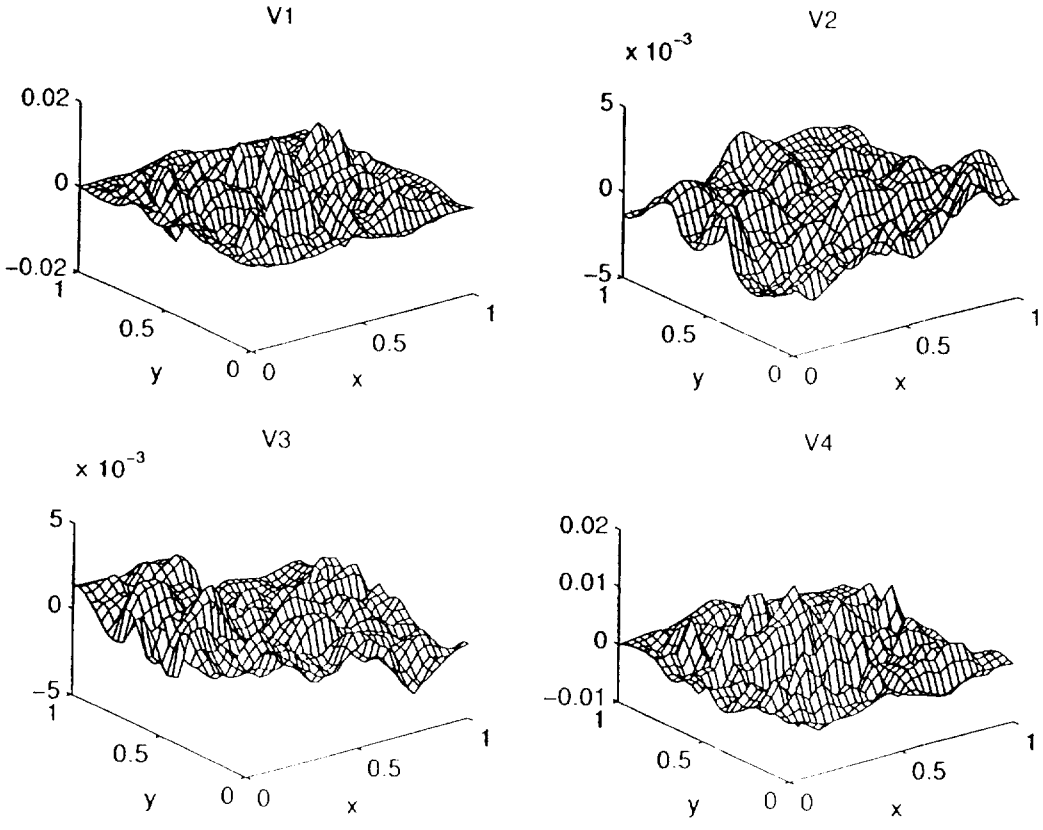


FIGURE 1. Solutions after 20 V(1,0)-cycles based on $G_1$

cycles for $h = \frac{1}{32}$ and $\lambda = 10$. Since $\mathbf{f} = \mathbf{0}$, the solutions are exactly errors so that we see the shape of errors and magnitudes componentwise. In Figure 1, even after 20 MG-cycles the multigrid method for FOSLS based on $G_1$ produces the errors which have both smoothing and highly oscillating parts. From Figure 2, we know that the multigrid method for FOSLS based on $G_2$ has errors which have only smoothing part. From the convergence rate and the shape of errors, we may think multigrid method based on $G_2$ is better than $G_1$. As a final conclusion, we need to develop a new FOSLS formulation to get rid of the effect of coupled boundary conditions between vector components.

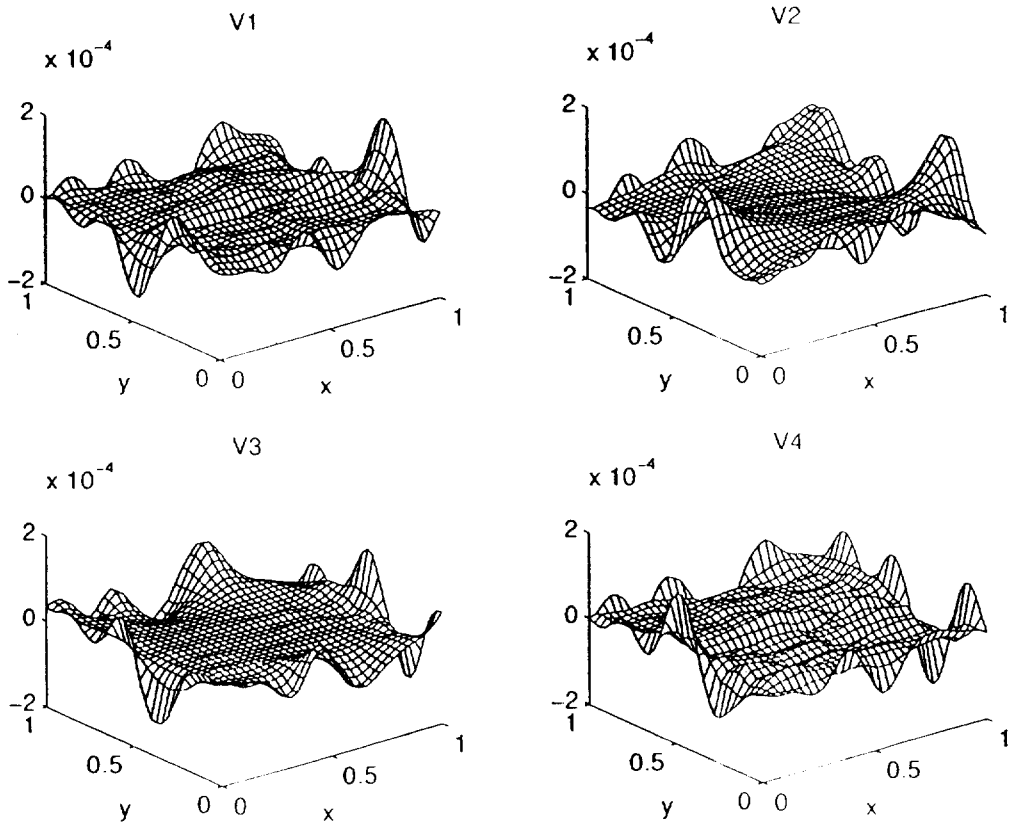


FIGURE 2. Solutions after 20 V(1,0)-cycles based on $G_2$

## References

[1] I. Babuška and B. Szabo, *On the rates of convergence of the finite element method*, Internat. J. Numer. Meth. Engng. **18** (1982), 323–341.

[2] S. C. Brenner, *A nonconforming mixed multigrid method for the pure traction problem in planar linear elasticity*, Math. Comp. **63** (1994), 435–460 & S1–S5.

[3] Z. Cai, T. A. Manteuffel and S. F. McCormick, *First-order system least squares for second-order partial differential equations: Part II*, SIAM J. Numer. Anal. **34** (1997), 425-454.

[4] ———, *First-order system least squares for the Stokes equations, with application to linear elasticity*, preprint.

[5] Z. Cai, T. A. Manteuffel, S. F. McCormick and S. Parter, *First-order system*

*least squares for the pure traction problem in planar linear elasticity*, preprint.

[6] P. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.

[7] J. Douglas and J. Wang, *An absolutely stabilized finite element method for the Stokes problem*, Math. Comp. **52** (1989), 495–508.

[8] P. Grisvard, *Problémes aux limits dans les polygones. Mode d'emploi*, EDF Bulletin de la Direction des Etudes et Recherches Series C Mathematiques, Informatique **1** (1986), 21–59.

[9] ———, *Singularités en elasticité*, Arch. Rat. Mech. Anal. **107** (1989), 157–180.

[10] R. B. Kellogg and J. E. Osborn, *A regularity results for the Stokes problem in a convex polygon*, J. Funct. Anal., **21** (1976), 397–431.

[11] O. A. Ladyhenskaya, *The Mathematical theory of viscous incompressible flows*, Gordon and Breach, New York, 1963.

[12] C.-O. Lee, *A conforming mixed finite element method for the pure*, to appear in Appl. Math. Comp.

[13] C.-O. Lee, T. A. Manteuffel and S. F. McCormick, private communication.

[14] S. McCormick,, *Multigrid methods for variational problems: Further results*, SIAM J. Numer. Anal. **21** (1984), 255-263.

[15] R. Stenberg, *A family of mixed finite elements for the elasticity problem*, Numer. Math. **53** (1988), 513-538.

[16] M. Vogelius, *An analysis of the p-version of the finite element method for nearly incompressible materials. Uniformly valid, optimal error estimates*, Numer. Math. **41** (1983), 39–53.

Department of Mathematics
Inha University
Inchon 402-751, Korea
*E-mail*: colee@math.inha.ac.kr