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A WEAK NEGATIVE ORTHANT DEPENDENCE

KWANG-HEE HAN

ABSTRACT. In this paper we introduce a new concept of negative dependence of multivariate random variables. This concept is weaker than the negative orthant dependence(NOD) but it enjoys some properties and preservation results of NOD

1. Introduction

Lehmann(1966) introduced the concept of negative quadrant dependence(NQD) together with some other dependence concepts. Ebrahimi and Ghosh(1981) extended this negative concepts of bivariate random variables to the multivariate random variables and introduced the concept of negative orthant dependence(NOD), and Joag-Dev and Proschan(1983) introduced the concept of negative association. Recently, Kim and Seo (1995) also introduced some negative dependence concepts which are weaker than negative association but stronger than negative lower orthant dependence or negative upper orthant dependence and derived some relations among them.

Most of the dependence concepts introduced in the literature are stronger than the negative orthant dependence. In this paper we introduce a new concept of negative dependence of multivariate random variables. The importance of this concept of negative dependence lies in the fact that it is weaker than the negative orthant dependence(NOD) and it enjoys some properties and preservation results of NOD.

In section 2, some definitions, some new results of weak negative quadrant dependence, and preliminary results are given. In section 3, some

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preservation results are derived such as closure under convolution, mixture of a certain type, transformations, and limit in distribution.

2. Preliminaries

In this section we introduce some definitions and basic properties.

DEFINITION 2.1. (Ebrahimi and Ghosh(1981)) The random variables $X_1, ..., X_n$ are said to be negative upper orthant dependent(NUOD) if

$$P(X_1 > x_1, ..., X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i) \tag{2.1}$$

for all real numbers $x_1, ..., x_n$.

DEFINITION 2.2. (Ebrahimi and Ghosh(1981)) The random variables $X_1, ..., X_n$ are said to be negative lower orthant dependent(NLOD) if

$$P(X_1 \leq x_1, ..., X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i) \tag{2.2}$$

for all real numbers $x_1, ..., x_n$.

The random variables $X_1, ..., X_n$ are said to be negative orthant dependent(NOD) if they are NUOD and NLOD. For $n = 2$, (2.1) and (2.2) are equivalent. However, as one might expect, these are not equivalent for $n \geq 3$. Ebrahimi and Ghosh(1981) gave an example of trivariate distribution which is NUOD, but not NLOD.

DEFINITION 2.3. The random variables $X_1, ..., X_n$ are said to be weakly negative upper orthant dependent of the first type(denoted by WNUOD1) if

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \{P(\bigcap_{i=1}^{n} X_i > s_i) - \prod_{i=1}^{n} P(X_i > s_i)\} ds_n ... ds_1 \leq 0 \tag{2.3}$$

for all real numbers $x_1, ..., x_n$ and $X_1, ..., X_n$ are said to be weakly negative upper orthant dependent of the second type(denoted by WNUOD2) if

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \{P(\bigcap_{i=1}^{n} X_i > s_i) - \prod_{i=1}^{n} P(X_i > s_i)\} ds_n ... ds_1 \leq 0 \tag{2.4}$$

for all real numbers $x_1, ..., x_n$.

We call that the random variables $X_1, ..., X_n$ are weakly negative upper orthant dependent(denoted by WNUOD) if they are WNUOD1 and WNUOD2.

DEFINITION 2.4. The random variables $X_1, ..., X_n$ are said to be weakly negative lower orthant dependent of the first type(denoted by WNLOD1) if

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \{P(\bigcap_{i=1}^{n} X_i \leq s_i) - \prod_{i=1}^{n} P(X_i \leq s_i)\}\, ds_n ... ds_1 \leq \ 0 \tag{2.5}$$

for all real numbers $x_1, ..., x_n$ and $X_1, ..., X_n$ are said to be weakly negative lower orthant dependent of the second type(denoted by WNLOD2) if

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \{P(\bigcap_{i=1}^{n} X_i \leq s_i) - \prod_{i=1}^{n} P(X_i \leq s_i)\}\, ds_n ... ds_1 \ \leq \ 0 \tag{2.6}$$

for all real number $x_1, ..., x_n$.

We call that the random variables $X_1, ..., X_n$ are weakly negative lower orthant dependent(denoted by WNLOD) if they are WNLOD1 and WNLOD2. We also call that random variables $X_1, ..., X_n$ are weakly negative orthant dependent(denoted by WNOD) if they are WNUOD and WNLOD. In the bivariate case, (X, Y)(or the distribution H) is said to be weakly negative quadrant dependent of first type(denoted by WNQD1) if

$$\int_{x}^{\infty} \int_{y}^{\infty} [P(X > s, Y > t) - P(X > s)P(Y > t)] dtds \ \leq \ 0$$

or

$$\int_{x}^{\infty} \int_{y}^{\infty} [P(X \leq s, Y \leq t) - P(X \leq s)P(Y \leq t)]\, dtds \ \leq \ 0$$

and it is said to be weakly negative quadrant dependent of the second type(denoted by WNQD2) if

$$\int_{-\infty}^{x} \int_{-\infty}^{y} [P(X > s, Y > t) - P(X > s)P(Y > t)] dtds \ \leq \ 0$$

or

$$\int_{-\infty}^{x} \int_{-\infty}^{y} [P(X \leq s, Y \leq t) - P(X \leq s)P(Y \leq t)] dtds \ \leq \ 0.$$

We say that the bivariate random variable (X, Y) is weakly negative quadrant dependent(denoted by WNQD) if it is WNQD1 and WNQD2(see

Alzaid, (1990)).

It is obvious that if the random variables $X_1, ..., X_n$ are NOD then they are WNOD. But the following example shows that WNOD does not imply NUOD.

EXAMPLE 2.5. Let X_1, X_2, X_3 be random variables with the following joint probabilities $P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$:

		$X_3 = 0$			$X_3 = 1$		
		X_2			X_2		
		0	-1	-2	0	-1	-2
X_1	0	2/40	0	3/40	2/40	0	3/40
	1	6/40	3/40	1/40	6/40	3/40	1/40
	2	0	1/40	4/40	0	1/40	4/40

By tedious calculation it can be checked that (X_1, X_2, X_3) is WNUOD1 but not NUOD.

We list below a number of basic properties of WNUOD and WNLOD variables :

(N0) Any set of NUOD(NLOD) random variables is WNUOD(WNLOD).

(N1) Any subset of WNUOD(WNLOD) random variables of size ≥ 2 is WNUOD(WNLOD).

(N2) If $X_1, ..., X_n$ are WNUOD(WNLOD) then $a_1X_1 + b_1, ..., a_nX_n + b_n$ is WNUOD(WNLOD) for all $a_i > 0$, $i = 1, 2, ..., n$.

(N3) The union of independent sets of WNUOD(WNLOD) random variables is WNUOD(WNLOD).

DEFINITION 2.6. (Ebrahimi and Ghosh(1981)) A random vector $\underline{Y}$ is stochastically increasing(decreasing) in the random vector $\underline{X}$ if $E[f(\underline{Y}) \mid \underline{X} = \underline{x}]$ is increasing(decreasing) in $\underline{x}$ for all real valued increasing functions f. We shall use the abbreviations SI and SD for stochastically increasing and decreasing, respectively.

The following theorem gives a sufficient condition for WNQD-ness.

THEOREM 2.7. (Alzaid(1990)) (X, Y) *is WNQD1(WNQD2) if and only if*

$$Cov(f(X_1), g(X_2)) \geq 0$$

for all functions f and g such that f is increasing nonnegative convex (nonpositive concave) and g decreasing nonpositive concave (nonnegative convex).

THEOREM 2.8. *Let*
(a) (X_1, X_2) given λ, a scalar random variable be conditionally WNQD1 (WNQD2), and
(b) X_1 be SI in λ, and X_2 be SD in λ, or
(b′) X_1 be SD in λ, and X_2 be SI in λ.
Then (X_1, X_2) is WNQD1(WNQD2).

PROOF. Let f be an increasing nonnegative convex(nopositive concave) function and g be a decreasing nonpositive concave(nonnegative convex) function.

(2.7)
$$Cov[f(X_1), g(X_2)] = Cov(E[f(X_1)|\lambda], E[g(X_2)|\lambda]) + E[Cov(f(X_1, g(X_2)|\lambda)]$$

From Definition 2.5 the conditional expections in the first term on the right hand side of (2.7) are increasing function of λ. By property P3(Esary et al.(1967)) λ is associated. Thus by Esary et al.(1967) the covariance of conditional expections in the first term is nonnegative. Since conditioned on λ, (X_1, X_2) is WNQD1(WNQD2) by Theorem 2.6 the second term on the right hand side of (2.7) is also nonnegative. It follows that $Cov(f(X_1), g(X_2)) \geq 0$. Thus (X_1, X_2) is WNQD1(WNQD2). □

COROLLARY 2.9. *Let (X_1, X_2) be WNQD1(WNQD2) and let Z be independent of (X_1, X_2). Define $X = X_1 + aZ$, $Y = X_2 + bZ$. If $ab \leq 0$ then (X, Y) is WNQD1(WNQD2).*

PROOF. Let $a \geq 0$, $b \leq 0$. Then $X = X_1 + aZ$ is SI in Z and $Y = X_2 + bZ$ is SD in Z. Since (X, Y), given Z is WNQD1(WNQD2), by Theorem 2.7, (X, Y) is WNQD1(WNQD2). Similarly, one handles the case $a \leq 0$, $b \geq 0$. □

We close this section by introducing orthant convex order and orthant concave order notions.

DEFINITION 2.10. (Shaked, Shanthikumar(1994)) For n-dimensional random vectors $\underline{X} = (X_1, ..., X_n)$ and $\underline{Y} = (Y_1, ..., Y_n)$, $\underline{X}$ is smaller than $\underline{Y}$

in the upper orthant-convex order(denoted by $\underline{X} \leq_{uo-cx} \underline{Y}$) if and only if

$$\int_{x_1}^{\infty}\int_{x_2}^{\infty}\cdots\int_{x_n}^{\infty} P(X_1 > s_1, X_2 > s_2, ..., X_n > s_n)ds_n...ds_2ds_1$$
$$\leq \int_{x_1}^{\infty}\int_{x_2}^{\infty}\cdots\int_{x_n}^{\infty} P(Y_1 > s_1, Y_2 > s_2, ..., Y_n > s_n)ds_n...ds_2ds_1$$

for all $\underline{x} = (x_1, ..., x_n)$, and $\underline{X}$ is smaller than $\underline{Y}$ in the lower orthant-concave order(denoted by $\underline{X} \leq_{lo-cv} \underline{Y}$ if and only if

$$\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} P(X_1 \leq s_1, X_2 \leq s_2, ..., X_n \leq s_n)ds_n...ds_2ds_1$$
$$\leq \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} P(Y_1 \leq s_1, Y_2 \leq s_2, ..., Y_n \leq s_n)ds_n...ds_2ds_1$$

for all $\underline{x} = (x_1, ..., x_n)$.

According to Alzaid(1990) the order $\leq_{uo-cx}$ is an extension of the variability order(see e.g. Ross(1983) p270), which is also called convex order(see e.g. Stoyan(1983) p8) and the order $\leq_{lo-cv}$ is an extension of concave order (see e.g. Stoyan(1983) p11).

THEOREM 2.11. (Shaked, Shanthikumar(1994)) *Let* $\underline{X} = (X_1, ..., X_n)$ *and* $\underline{Y} = (Y_1, ..., Y_n)$ *be n-dimensional random vectors. Then*
(i) $\underline{X} \leq_{uo-cx} \underline{Y}$ *if and only if*

$$E[\prod_{i=1}^{n} g_i(X_i)] \leq E[\prod_{i=1}^{n} g_i(Y_i)]$$

for all nonnegative increasing convex functions $g_1, ..., g_n$.
(ii) $\underline{X} \leq_{lo-cv} \underline{Y}$ *if and only if*

$$E[\prod_{i=1}^{n} h_i(X_i)] \leq E[\prod_{i=1}^{n} h_i(Y_i)]$$

for all nonnegative increasing concave functions $h_1, ..., h_n$.

3. Some preservation results with application

DEFINITION 3.1. A random vector $\underline{Y}$ is said to be stochastically right tail increasing(decreasing) in the random vector $\underline{X}$ if $E\ [f(\underline{Y}) \mid \underline{X} > \underline{x}]$ is increasing(decreasing) in $\underline{x}$ for every real valued increasing function f.

THEOREM 3.2. *Let*
(a) $\underline{X} = (X_1, ..., X_n)$ *be WNUOD,*
(b) $Y_1, ..., Y_m$ *be conditionally independent given* $\underline{X}$, *and*
(c) Y_i *be stocastically right tail decreasing in* $\underline{X}$ *for all* $i = 1, ..., m$.
Then (i) $(\underline{X}, \underline{Y})$ *is WNUOD,* (ii) $\underline{Y}$ *is WNUOD.*

PROOF. (i) First we show the WNUOD1 case:

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} P(\bigcap_{i=1}^{n}(X_i > s_i), \bigcap_{i=1}^{m}(Y_j > t_j))dt_m...dt_1 ds_n...ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} P(\bigcap_{j=1}^{m}(Y_j > t_j) | \bigcap_{i=1}^{n}(X_i > s_i)) P(\bigcap_{i=1}^{n}(X_i > s_i))dt_m...dt_1 ds_n...ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{j=1}^{m} P((Y_j > t_j) | \bigcap_{i=1}^{n}(X_i > s_i)) P(\bigcap_{i=1}^{n}(X_i > s_i))dt_m...dt_1 ds_n...ds_1$$

$$\leq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_m}^{\infty} \prod_{i=1}^{n} P((X_i > s_i) \prod_{j=1}^{m} P((Y_j > t_j)dt_m...dt_1 ds_n...ds_1.$$

The second equality follows from assumption (b) and the above inequality follows from assumptions (a) and (c). Similarly, the WNUOD2 case is proved.

(ii) follows from (i) immediately by property (N1). □

THEOREM 3.3. *Let* H_0 *and* H_1 *be two multivariate WNUOD distributions both having the same one dimensional marginals. Then if* $\bar{H}_\alpha = \alpha\bar{H}_0 + (1-\alpha)\bar{H}_1$, $\alpha \in (0,1)$, *where* $\bar{H}_\alpha(x_i, ..., x_n) = P_{H_\alpha}(X_1 > x_1, ..., X_n > x_n)$, $\bar{H}_i(x_1, ..., x_n) = P_{H_i}(X_1 > x_1, ..., X_n > x_n)$, $i = 0, 1$, *then* $\bar{H}_\alpha$ *is also WNUOD.*

PROOF. By definition, the one dimensional marginals of $\bar{H}_\alpha$ are the same as those of $\bar{H}_0$ or $\bar{H}_1$. Next,

$$\begin{aligned}
\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} & P_{\bar{H}_\alpha}(X_1 > s_1, \ldots, X_n > s_n) ds_n \ldots ds_1 \\
&= \alpha \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P_{\bar{H}_0}(X_1 > s_1, \ldots, X_n > s_n) ds_n \ldots ds_1 \\
&\quad + (1-\alpha) \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P_{\bar{H}_1}(X_1 > s_1, \ldots, X_n > s_n) ds_n \ldots ds_1 \\
&\leq \alpha \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(X_i > s_i) ds_n \ldots ds_1 \\
&\quad + (1-\alpha) \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(X_i > s_i) ds_n \ldots ds_1 \\
&= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(X_i > s_i) ds_n \ldots ds_1
\end{aligned}$$

Hence H_α is WNUOD1. Similarly, H_α is WNUOD2 and thus the proof is complete. □

COROLLARY 3.4. *Let H_0 and H_1 be WNLOD distributions both having the same marginals. Put $H_\alpha = \alpha H_0 + (1-\alpha)H_1, 0 < \alpha < 1$. Then H_α is also WNLOD.*

LEMMA 3.5. *Let $\underline{X} = (X_1, \ldots, X_n)$ be a random vector and let $\underline{X}^* = (X_1^*, X_2^*, \ldots, X_n^*)$ be an n-dimensional random vector with independent components such that $X_i^* =^d X_i$, $i = 1, \ldots, n$ (where $=^d$ stands for equality in distribution). Then*

(i) $\underline{X}$ is WNUOD1 if and only if $\underline{X} \leq_{uo-cx} \underline{X}^*$,

(ii) $\underline{X}$ is WNLOD2 if and only if $\underline{X} \leq_{lo-cv} \underline{X}^*$.

PROOF. (i) ($\Rightarrow$) Assume $\underline{X}$ is WNUOD1. Then

$$\begin{aligned}
\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} & P(X_1 > s_1, \ldots, X_n > s_n) ds_n \ldots ds_1 \\
&\leq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(X_i > s_i) ds_n \ldots ds_1
\end{aligned}$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \prod_{i=1}^{n} P(X_i^* > s_i) ds_n \ldots ds_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(X_1^* > s_1, \ldots, X_n^* > s_n) ds_n \ldots ds_1.$$

Hence $\underline{X} \leq_{uo-cx} \underline{X^*}$.
($\Leftarrow$) It follows from assumptions that

$$\begin{aligned} \int_{x_1}^{\infty} & \cdots \int_{x_n}^{\infty} [P(X_1 > s_1, \ldots, X_n > s_n) - \prod_{i=1}^{n} P(X_i > s_i)] ds_n \ldots ds_1 \\ & \leq \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} [P(X_1^* > s_1, \ldots, X_n^* > s_n) - \prod_{i=1}^{n} P(X_i^* > s_i)] ds_n \ldots ds_1 \\ & = 0. \end{aligned}$$

From assumption that $\underline{X^*}$ has independent components in the right hand side zero follows. Hence $\underline{X}$ is WNUOD1. Similarly, the proof of (ii) is obtained. □

From Theorem 2.10 and Lemma 3.5 we obtain following theorem.

THEOREM 3.6. *Let $\underline{X} = (X_1, \ldots, X_n)$ be a random vector.*
(i) *$\underline{X} = (X_1, \ldots, X_n)$ is WNUOD1 if and only if*

$$E[\prod_{i=1}^{n} f_i(X_i)] \leq \prod_{i=1}^{n} E[f_i(X_i)]$$

for all increasing nonnegative convex functions $f_1, \ldots, f_n$.
(ii) *$\underline{X} = (X_1, \ldots, X_n)$ is WNLOD2 if and only if*

$$E[\prod_{i=1}^{n} h_i(X_i)] \leq \prod_{i=1}^{n} E[h_i(X_i)]$$

for all increasing nonnegative concave functions $h_1, \ldots, h_n$.

THEOREM 3.7. (i) *$\underline{X} = (X_1, \ldots, X_n)$ is WNUOD1 if and only if $(f_1(X_1), \ldots, f_n(X_n))$ is WNUOD1 for all increasing convex functions $f_1, \ldots, f_n$.*
(ii) *$\underline{X} = (X_1, \ldots, X_n)$ is WNLOD2 if and only if $(g_1(X_1), \ldots, g_n(X_n))$ is WNLOD2 for all increasing concave functions $g_1, \ldots, g_n$.*

PROOF. It is sufficient to show only if part. (i) Assume $\underline{X} = (X_1, ..., X_n)$ is WNUOD1. Then for all increasing nonnegative convex funcions $F_1, ..., F_n$,

$$E[\prod_{i=1}^{n} F_i(f_i(X_i))] \leq \prod_{i=1}^{n} E[F_i(f_i(X_i))]$$

since $F_i \cdot f_i$'s are increasing nonnegative convex funcions. Hence $(f_1(X_1), ..., f_n(X_n))$ is WNUOD1 according to Theorem 3.6.

(ii) Assume $\underline{X} = (X_1, ..., X_n)$ is WNLOD2. Then for all increasing nonnegative concave funcions $G_1, ..., G_n$,

$$E[\prod_{i=1}^{n} G_i(g_i(X_i))] \leq \prod_{i=1}^{n} E[G_i(g_i(X_i))]$$

since $G_i \cdot g_i$'s are increasing nonnegative concave funcions. Hence $(g_1(X_1), ..., g_n(X_n))$ is NLOD2 according to Theorem 3.6. □

The next theorem demonstrates the preservation of the WNOD property under limits.

THEOREM 3.8. *Let $\{\underline{X}_n, n \geq 1\}$ be a sequence of WNOD p-dimensional random vectors with distribution function H_n such that $H_n \to H$ weakly as $n \to \infty$, where H is the distribution function of a random vector $\underline{X} = (X_1, ..., X_p)$. Then $\underline{X}$ is WNOD.*

PROOF. We will only show the WNUOD case. Writing $\underline{X}_n = (X_{1n}, ..., X_{pn})$, $n \geq 1$, $\underline{X} = (X_1, ..., X_p)$ for any real $x_1, ..., x_p$, we have by assumptions,

$$\begin{aligned} &\int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} P(X_1 > s_1, ..., X_p > s_p) ds_p ... ds_1 \\ &= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} [\lim_{n\to\infty} P(X_{1n} > s_1, ..., X_{pn} > s_p)] ds_p ... ds_1 \\ &\leq \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} [\lim_{n\to\infty} \prod_{j=1}^{p} P(X_{jn} > s_j)] ds_p ... ds_1 \\ &= \int_{x_1}^{\infty} \cdots \int_{x_p}^{\infty} [\prod_{j=1}^{p} P(X_j > s_j)] ds_p ... ds_1. \end{aligned}$$

Thus $\underline{X}$ is WNUOD1. Similarly, $\underline{X}$ is WNUOD2. The proof is complete. □

THEOREM 3.9. *Let $(X_{11},...X_{1p}),...,(X_{n1},...X_{np})$ be independent random vectors and let $Y_1 = f_1(X_{11},...,X_{n1}),...,Y_p = f_p(X_{1p},...,X_{np})$. Assume that for each i, $(X_{i1},...,X_{ip})$ is WNUOD1 and $f_1,...,f_p$ are nonnegative increasing convex functions for the ith coordinate, $i = 1,...,n$, then $(Y_1,...,Y_p)$ is WNUOD1.*

PROOF. Define for $k = 2,...,n$, $j = 1,...,p$,

$$h_j^{(k)}(X_{(k+1)j},...,X_{nj}) = E\{h_j^{(k-1)}(X_{kj},...,X_{nj})|X_{(k+1)j},...,X_{nj}\}. \tag{3.1}$$

We also define $h_j^{(1)}(X_{2j},...,X_{nj}) = E\{h_j(X_{1j},...,X_{nj})|X_{2j},...,X_{nj}\}$ for any function $h_j(X_{1j},...,X_{nj})$ having property of f_i. Then we obtain

$$\begin{aligned} Eh_j(X_{1j},...,X_{nj}) &= Eh_j^{(1)}(X_{2j},...,X_{nj} \\ &= Eh_j^{(2)}(X_{3j},...,X_{nj}) \\ &\vdots \\ &= Eh_j^{(n-1)}(X_nj). \end{aligned} \tag{3.2}$$

In view of Theorem 3.6 it is sufficient to prove that for any functions $h_1,...,h_p$ having properties of $f_1,...,f_p$, respectively,

$$E[\textstyle\prod_{j=1}^{p}(h_j(X_{1j},...,X_{nj}))] \le \prod_{j=1}^{p} E[(h_j(X_{1j},...,X_{nj}))] \tag{3.3}$$

This is so since for any nonnegative increasing convex functions $k_1,...,k_p$ the functions $k_1f_1,...,k_pf_p$ have the same properties as do $f_1,...,f_p$. To show (3.3) is valid, we follow an iteration argument.

$$\begin{aligned} E[\prod_{j=1}^{p}(h_j(X_{1j}, \quad ... \quad ,X_{nj}))] &= E[E[\prod_{j=1}^{p}(h_j(X_{1j},...,X_{nj})|X_{2j},...,X_{nj})]] \\ &\le E[\prod_{j=1}^{p} E[(h_j(X_{1j},...,X_{nj})|X_{2j},...,X_{nj})]] \end{aligned}$$

$$= E[\prod_{j=1}^{p} h_j^{(1)}(X_{2j}, ..., X_{nj})]$$

(by proceeding with the iteration argument used above)

$$= E[\prod_{j=1}^{p} h_j^{(n-1)}(X_{nj})] \leq \prod_{j=1}^{p} E[h_j^{(n-1)}(X_{nj})]$$

$$= \prod_{j=1}^{p} E[h_j(X_{1j}, ..., X_{nj})].$$

Note that since $(X_{i1}, ..., X_{ip})$ is WNUOD1 and h_j's are nonnegative increasing convex functions the first above inequality holds. From the fact that $(X_{n1}, ..., X_{np})$ is WNUOD1 and $h_j^{(n-1)}$'s are nonnegative increasing convex functions the second above inequality follows and the last inequality follows from (3.2). The proof is complete. □

A similar result holds for the WNUOD2 property.

COROLLARY 3.10. *Let $(U_1, ..., U_p)$ be independent and independent of $(X_{11}, ..., X_{1p})$, ..., $(X_{n1}, ..., X_{np})$ and let $Y_1 = f_1(U_1, X_{11}, ..., X_{n1}), ..., Y_p = f_p(U_p, X_{1p}, ..., X_{np})$. Assume that for each i, $(X_{i1}, ..., X_{ip})$ is WNUOD1 and $f_1, ..., f_p$ are nonnegative increasing convex functions for each coordinate, then $(Y_1, ..., Y_p)$ is WNUOD1.*

PROOF. This follows from Theorem 3.9 since for each i $(U_i, X_{i1}, ..., X_{ip})$ is WNUOD1. □

The followig Theorem is an application of Theorem 3.9 which is very important in recognizing WNUOD1 in compound distributions which arise naturally in stochastic process.

THEOREM 3.11. *Let $(N_1, ..., N_p)$ be a p-variate variable with components assuming values in the set $\{1, 2, ...\}$ and let $\{(X_{i1}, ..., X_{ip}) : i \geq 1\}$ be a sequence of nonnegative independent p-variate random variables independent of $(N_1, ..., N_p)$. Suppose that $(N_1, ..., N_p)$ is WNUOD1 and that $(X_{i1}, ..., X_{ip})$ is WNUOD1. Define $\underline{Y} = (Y_1, ..., Y_p)$ by $Y_i = \sum_{j=1}^{N_i} X_{ji}$, $i = 1, ..., p$. Then $\underline{Y} = (Y_1, ..., Y_p)$ is WNUOD1.*

PROOF. Let $f_1, ..., f_p$ be nonnegative increasing convex functions. Then

$$\begin{aligned}
E[\prod_{i=1}^{p} f_i(Y_i)] &= E[E[\prod_{i=1}^{p}\{f_i(\sum_{j=1}^{N_i} X_{ji})|N_i = n_i\}]] \\
&= E[E[\{\prod_{i=1}^{p} f_i(\sum_{j=1}^{n_i} X_{ji})\}]] \\
&\leq E[\prod_{i=1}^{p} E[\{f_i(\sum_{j=1}^{n_i} X_{ji})\}]] \\
&\leq \prod_{i=1}^{p} E[E[\{f_i(\sum_{j=1}^{n_i} X_{ji})\}]] \\
&= \prod_{i=1}^{p} E[E[\{f_i(\sum_{j=1}^{N_i} X_{ji})|N_i = n_i\}]] \\
&= \prod_{i=1}^{p} E[\{f_i(Y_i)\}].
\end{aligned}$$

The first inequality follows from Theorem 3.9 and assumption and the second inequality follows from the fact that $E[f_i(\sum_{j=1}^{n_i} X_{ji})]$'s are increasing convex functions in n_i. This completes the proof. □

EXAMPLE 3.12. *Let $\{N_1(t), ..., N_p(t) : t \geq 0\}$ be the p-variate Poisson processes, i.e., $N_1(t) = Z_1(t) + W(t), ..., N_p(t) = Z_p(t) + W(t)$ where $Z_1(t), ..., Z_p(t)$ and $W(t)$ are independent Poisson processes. Let $\{(X_{n1}, ..., X_{np}) : n = 0, 1, 2, ...\}$ be a sequence of independent and identically distributed random variables. Define the p-variate compound Poisson process $\{Y_1(t), ..., Y_p(t) : t \geq 0\}$ by*

$$Y_1(t) = \sum_{n=0}^{N_1(t)} X_{n1}, ..., Y_p(t) = \sum_{n=0}^{N_p(t)} X_{np}.$$

Since $\{N_1(t), ..., N_p(t)\}$ is WNUOD1 for every $t \geq 0$, consequently an application of Theorem 3.11 implies $\{Y_1(t), ..., Y_p(t)\}$ is WNUOD1 for every $t \geq 0$ whenever $(X_{n1}, ..., X_{np})$ is WNUOD1.

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Departments of Computer Sciences
ChonBuk San Up University
727, Wolha-Ri, Impi, Kunsan
Chonbuk 573-930, Korea